PROJECTIVITY AND BIRATIONAL GEOMETRY OF BRIDGELAND MODULI SPACES

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ABSTRACT. We construct a family of nef divisor classes on every moduli space of stable complexes in the sense of Bridgeland. This divisor class varies naturally with the Bridgeland stability condition. For a generic stability condition on a K3 surface, we prove that this class is ample, thereby generalizing a result of Minamide, Yanagida, and Yoshioka. Our result also gives a systematic explanation of the relation between wall-crossing for Bridgeland-stability and the minimal model program for the moduli space.

We give three applications of our method for classical moduli spaces of sheaves on a K3 surface:

1. We obtain a region in the ample cone in the moduli space of Gieseker-stable sheaves only depending on the lattice of the K3.

2. We determine the nef cone of the Hilbert scheme of n points on a K3 surface of Picard rank one when n is large compared to the genus.

3. We verify the "Hassett-Tschinkel/Huybrechts/Sawon" conjecture on the existence of a birational Lagrangian fibration for the Hilbert scheme in a new family of cases.

CONTENTS

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1. INTRODUCTION

In this paper, we give a canonical construction of determinant line bundles for moduli spaces of Bridgeland-semistable objects. Our construction has two advantages over the classical construction for semistable sheaves: our divisor class varies naturally with the stability condition, and we can show—with a purely categorical proof, based on results by Abramovich and Polishchuk—that our divisor is automatically nef. This leads to more precise results even in the case when the moduli space coincides with a classical moduli space of Gieseker-stable sheaves. In the case of K3 surfaces, we use it to prove projectivity of the moduli space, which generalizes a recent result of Minamide, Yanagida, and Yoshioka. Our construction also explains a picture envisioned by Bridgeland, and observed in examples by Arcara-Bertram and others, that relates wall-crossing under a change of stability condition to the birational geometry and the minimal model program of the moduli space.

Moduli spaces of complexes. The importance of moduli spaces of complexes first appeared in Bridgeland's flop construction: in [Bri02, VdB04], it is shown that a flop of a smooth threefold can be interpreted as a moduli space parameterizing "perverse ideal sheaves" in the bounded derived category of coherent sheaves. More recently, moduli spaces of complexes have turned out extremely useful in Donaldson-Thomas theory; see [Tod11] for a survey and [PT09, JS08, KS08, Bay09, Tod09, Tod10, Bri11] for related results.

In the ideal situation, the necessary notion of stability of complexes can be given in terms of Bridgeland's notion of a stability condition on the derived category, introduced in [Bri07]. One of the advantages of Bridgeland's notion of stability over other constructions (as in [Tod09, Bay09]) is that the space of Bridgeland stability conditions admits a well-behaved wall and chamber structure which controls when a variation of the stability condition changes the moduli space of stable objects with given invariants. However, unlike in the case of Simpson- or Gieseker-stability for sheaves, Bridgeland's notion does not appear to be connected to a GIT problem in general. As a consequence, while there are well-established methods to prove the existence of moduli spaces as Artin stacks, or as algebraic spaces (due to the work in [Ina02, Lie06, Tod08, AP06]), there are so far only ad-hoc methods to prove projectivity of moduli spaces, or to construct coarse moduli spaces.

In this paper, we propose a solution to this problem by constructing a family of numerically positive divisor classes on any moduli space of Bridgeland-stable complexes. While our approach is very general, we concentrate on the case of K3 surfaces. In this situation, Bridgeland has given an explicit description of (a connected component of) the space of stability conditions in [Bri08], and Toda has proved existence of moduli stacks of semistable objects as Artin stacks of finite type over $\mathbb{C}$ in [Tod08]. In Section 16 of the arXiv-version of [Bri08], Bridgeland proposed the following conjecture:

Conjecture 1.1 (Bridgeland)**.** *Given a stability condition σ, and a numerical equivalence class v, there exists a coarse moduli space $M_\sigma(v)$ of complexes of numerical type v which are σ-semistable. Changing the stability condition will produce birational morphisms between these coarse moduli spaces.*

Our results give a partial proof of this conjecture. They give a close relation between walls in the space of stability conditions, and walls in the movable cone of the moduli space separating the ample cones of different birational models.

A family of nef divisors on Bridgeland-moduli spaces. Let X be a smooth projective variety over the complex numbers. We denote by $\mathrm{D^b}(X)$ the bounded derived category of coherent sheaves on X and by $\mathrm{Stab}(X)$ the space of Bridgeland stability conditions on $\mathrm{D^b}(X)$. (In Section 2 we recall the basic properties of Bridgeland stability conditions.)

Given a stability condition $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$ and a choice of numerical invariants v, assume that we are given a family of σ-semistable objects of class v parametrized by S (for example, S could be the fine moduli space $M_\sigma(v)$ of stable objects, if that exists). Let $\mathcal{E} \in \mathrm{D^b}(S \times X)$ be a universal family. Then we can define a class $\ell_\sigma(v) \in N^1(S)$ as follows: To every curve $C \subset S$, we associate

$$\ell_\sigma \colon [C] \mapsto \ell_\sigma.C := \Im\left(-\frac{Z(\Phi_{\mathcal{E}} O_C))}{Z(v)}\right), \tag{1}$$

where $\Phi_{\mathcal{E}} \colon \mathrm{D^b}(S) \to \mathrm{D^b}(X)$ is the Fourier-Mukai functor with kernel $\mathcal{E}$, and $\mathcal{O}_C$ is the structure sheaf of C. It is easy to prove that ℓ_σ defines a numerical divisor class $\ell \in \mathrm{NS}(S)$. Our main result, the Positivity Lemma 3.3, implies:

Theorem 1.2. *The divisor class ℓ_σ is nef. Further, for a curve C, we have $\ell_\sigma.C = 0$ if and only if for two general closed points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent.*

(Recall that two objects semistable are S-equivalent if their Jordan-Hölder filtrations into stable factors of the same phase have the same stable factors, up to reordering.) While our class ℓ_σ can also be given as a determinant line bundle, our construction avoids any additional choices: it depends only on the choice of a stability condition. It's main advantage is that we can show its positivity property directly, without having to use GIT constructions as for example in [LP92]. Further, when we vary σ within a chamber of $\mathrm{Stab}(X)$ for which $M_\sigma(v)$ remains constant, we obtain a family of nef divisors. In examples, this family covers the entire nef cone of $M_\sigma(v)$.

The proof of Theorem 1.2 will take the whole Section 3. The basic ingredient is a construction by Abramovich of Polishchuk in [AP06, Pol07]: Given the t-structure on X associated to the Bridgeland stability condition, their construction produces a t-structure on $\mathrm{D^b}(S \times X)$, for any scheme S. This categorical ingredient allows us to transfer the basic positivity of the "central charge", see equation (3), to the positivity of ℓ_σ as a divisor.

Chambers in $\mathrm{Stab}(X)$ and the nef cone of the moduli spaces. Consider a chamber $\mathcal{C}$ for the wall-and-chamber decomposition with respect to v; then the moduli space $M_\sigma(v)$ (if it exists) is constant for $\sigma \in \mathcal{C}$. We thus obtain an essentially linear map

$$l \colon \overline{\mathcal{C}} \to \mathrm{Nef}(M_\sigma(v)). \tag{2}$$

This immediately begs for the following two questions:

Question 1: Do we actually have $l(\mathcal{C}) \subset \mathrm{Amp}(M_{\mathcal{C}}(v))$?
Question 2: What will happen at the walls of $\mathcal{C}$?

In the case of K3 surfaces, we give a complete answer to the first question, and an almost-complete answer to the second question.

K3 surfaces: Projectivity of the moduli spaces. We now restrict to the case where X is a smooth projective K3 surface. In [MYY11a, MYY11b], the authors use a beautiful observation to reinterpret any moduli space of Bridgeland-semistable complexes on a K3 surface with Picard rank one as a moduli space of Gieseker-semistable sheaves on a Fourier-Mukai partner Y of X. Using an extension of their idea, we show that ℓ_σ gives an ample divisor on the moduli space:

Theorem 1.3. *Let X be a smooth projective K3 surface, and let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$. Assume that we are given a stability condition which is "generic" with respect to v. Then:*

(a) There exists a coarse moduli space $M_\sigma(v)$ of σ-semistable objects with Mukai vector v. It is a normal projective irreducible variety with $\mathbb{Q}$-factorial singularities.
(b) The class $\ell_\sigma(v)$ induces an ample divisor class on $M_\sigma(v)$.

This generalizes [MYY11b, Theorem 0.0.2], which shows projectivity of $M_\sigma(v)$ in the case where X has Picard rank one. The stability condition σ is "generic" if it does not lie on a wall in the wall and chamber decomposition of $\mathrm{Stab}(X)$ with respect to v.

K3 surfaces: Wall-crossing and birational geometry of the moduli spaces. One can hope to use Theorem 1.2 to understand how the moduli space changes when we vary the stability condition σ, i.e., to understand wall-crossing. The first step is to understand what happens when σ is not generic; we consider this in the case where v is a primitive class.

Let W be a wall of the chamber decomposition for v. Let σ_0 be a generic point of W, and let σ_+, σ_- be two generic stability conditions inside the two chambers adjacent to σ_0. By Theorem 1.3, both moduli spaces $M_\pm := M_{\sigma_\pm}(v)$ are non-empty irreducible symplectic projective manifolds. Since being semistable is a closed condition in $\mathrm{Stab}(X)$, the universal families $\mathcal{E}_\pm$ on $M_\pm$ are in particular families of σ_0-semistable objects. Theorem 1.2 also applies in this situation, and thus σ_0 produces nef divisor classes $\ell_{0,\pm}$ on $M_\pm$. In Section 7, we prove:

Theorem 1.4. *Let X be a smooth projective K3 surface, and let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive vector.*

(a) The divisor classes $\ell_{0,\pm}$ are big and nef, and they induce contraction morphisms

$$\pi_{\sigma_\pm} \colon M_\pm \to Y_\pm,$$

where $Y_\pm$ are normal irreducible projective varieties.

(b) If there exist σ_0-stable objects, and if their complement in $M_\pm$ has codimension at least two 2, then there are the following two possibilities:

- *If either $\ell_{0,+}$ or $\ell_{0,-}$ is ample, then they are both ample, and the birational map*

  $$f_{\sigma_0} \colon M_+ \dashrightarrow M_-$$

  obtained by crossing the wall in σ_0 extends to an isomorphism.
- *If $\ell_{0,\pm}$ is not ample, then $f_{\sigma_0} \colon M_+ \dashrightarrow M_-$ is the flop induced by $\ell_{0,+}$. More precisely, we have a commutative diagram of birational maps*

  $$\begin{tikzcd} M_{\sigma_+}(v) \arrow[rr, dashed, "f_{\sigma_0}"] \arrow[dr, "\pi_+"'] & & M_{\sigma_-}(v) \arrow[dl, "\pi_-"] \\ & Y_+ = Y_- & \end{tikzcd},$$

  and $f^*_{\sigma_0}\ell_{0,-} = \ell_{0,+}$.

Note that our Theorem does not say anything about the cases where there are no σ_0-stable objects, or the case where their complement has codimension one.[1]

They key point is to show that $\ell_{0,\pm}$ is big; we reduce this to the case of stable sheaves, in which case it was proven by Yoshioka by deformation to the Hilbert scheme in [Yos01b]. In some examples (including those considered in [ABL07, MM11]), we can also show that $Y_\pm$ is a connected components of the coarse moduli space of σ_0-semistable objects of class v. It is also worth pointing out that the contraction morphism from the moduli space of Gieseker-stable sheaves to the Uhlenbeck moduli space of μ-semistable sheaves (see [Li93]) is a particular example of the contraction morphism π_{σ_+}; this was observed on the level of sets of semistable objects in [LQ11], and on the level of moduli spaces in the recent preprint [Lo12]. We study many more examples of wall-crossing behavior in terms of our picture in Sections 8 and 9.

Nef cones of moduli spaces of stable sheaves. Our Positivity Lemma and Theorem 1.2 can give more precise results on the nef cone of the moduli space even in the situation where $M_\sigma(v)$ is a classical moduli space of sheaves. We give two such example applications:

[1]Our only result on the latter case is Lemma 9.10, which shows that in this case, we have $l_{0,+} = l_{0,-}$ under the natural identification of the Néron-Severi groups of $M_{0,\pm}$.

- In **Corollary 8.14**, we determine a region of the ample cone of the moduli space of Gieseker-stable sheaves on X that depends only on the lattice of X.
- In **Proposition 9.3**, we determine the nef cone of the Hilbert scheme of n points on a K3 surface of Picard rank one and genus g for $n \geq \frac{g}{2}+1$. Our result shows in particular that [HT10, Conjecture 1.2] will need to be modified (see Remark 9.4).

The strength of our approach is that it can produce both nef divisors (by Theorem 1.2) and extremal curves of the Mori cone (as curves of S-equivalent objects on the wall) at the same time.

Lagrangian fibrations for the Hilbert scheme. Let X be a surface with $\operatorname{Pic} X = \mathbb{Z} \cdot H$, and of degree $d = \frac{1}{2}H^2$. We consider the Hilbert scheme $\operatorname{Hilb}^n(X)$ of n points on X. According to a long-standing conjecture, which appeared in print in articles by Hassett-Tschinkel [HT01], Huybrechts [GHJ03] and Sawon [Saw03], and had been proposed earlier (see [Ver10]), the Hilbert scheme admits a birational model that has a Lagrangian fibration if and only if $d = \frac{k^2}{h^2}(n-1)$ form some integers k, h. While the only if part is clear, the other direction has only been proved for $h = 1$, independently by Markushevich in [Mar06] and Sawon in [Saw07]. We also point out that the case $k = 1$ follows from a recent paper by Kimura and Yoshioka, see [KY11].[2]

We give a proof of the conjecture in the case $h = 2$, and interpret all birational models as moduli spaces of Bridgeland-stable objects: We denote by $\tilde{H} \subset \operatorname{Hilb}^n(X)$ the divisor of subschemes intersecting a given curve in the linear system $|H|$, and by $B \subset \operatorname{Hilb}^n(X)$ be the reduced divisor of non-reduced subschemes.

Theorem 9.8. *Let X be a K3 surface with $\operatorname{Pic} X = \mathbb{Z} \cdot H$ and $H^2 = 2d$. Assume that there is an odd integer k with $d = \frac{k^2}{4}(n-1)$ for some integer n. Then:*

(a) The movable cone $\operatorname{Mov}(\operatorname{Hilb}^n(X))$ is generated by $\tilde{H}$ and $2\tilde{H} - kB$.
(b) The morphism induced by $\tilde{H}$ is the Hilbert-to-Chow morphism, while the one induced by $2\tilde{H} - kB$ is a Lagrangian fibration.
(c) All minimal models for $\operatorname{Hilb}^n(X)$ arise as moduli spaces of stable objects in $\mathrm{D}^{\mathrm{b}}(X)$ and their birational transformations are induced by crossing a wall in $\operatorname{Stab}^\dagger(X)$.

Some relations to existing work. Our construction was inspired by the results of [ABCH12]. The authors studied wall-crossing for the Hilbert scheme of points on $\mathbb{P}^2$, and found a surprisingly direct relation between walls $\operatorname{Stab}(\mathbb{P}^2)$ and walls in the movable cone of the Hilbert scheme separating nef cones of different birational models.

[2] Their construction can be described in quite classical terms: in this case, a generic element $I_Y \in \operatorname{Hilb}^n(X)$ admits a unique map $\mathcal{O}(-h \cdot H) \hookrightarrow I_Y$; generically, the cokernel will be stable and thus an element of $M(0, h \cdot H, 1)$, which admits a Lagrangian fibration.

In their case, the variation of moduli spaces can also be seen as a variation of GIT parameters, via the classical monad construction. More precisely, stable complexes with respect to a Bridgeland stability condition can be seen as θ-stable representations of a Beilinson quiver for $\mathbb{P}^2$, in the sense of King [Kin94]. In this case, our divisor class ℓ_σ agrees with the ample divisor coming from the affine GIT construction of these moduli spaces. More generally, our family of nef divisors generalizes a construction by Craw and Ishii in [CI04] that produces a family of nef divisors on moduli spaces of θ-stable quiver representations.

The analogue of Theorem 1.3 for abelian surfaces has been proved in [MYY11b]; a different method to prove projectivity was established in [MM11, Mac12].

Our main result implies that knowing the precise location of walls in $\mathrm{Stab}(X)$ has immediate applications to the geometry of the nef cone and the movable cone of the moduli spaces of stable objects. Various authors have considered the geometry of walls in $\mathrm{Stab}(X)$ explicitly: first examples were found by Arcara and Bertram in [ABL07, AB11]; the case where X is an abelian surface has been studied by Minamide, Yanagida and Yoshioka in [MYY11a, MYY11b, YY12], and by Meachan and Maciocia in [MM11, Mac12]; Lo and Qin studied the case of arbitrary surfaces in [LQ11]. As an example, our Corollary 8.14 is a straightforward combination of our main result and the main result of Kawatani in [Kaw11]; the above-mentioned authors had all used similar methods.

Our construction, and specifically Corollary 8.14 may prove useful for Le Potier's Strange Duality Conjecture, see [LP05] and [MOY10]. While Le Potier's construction produces line bundles with sections, it is more difficult to show that these line bundles are nef; the Positivity Lemma can fill this gap.

It would be very interesting to relate our picture to results by Markman on the movable cone in [Mar11]. Markman proves that the closure of the movable cone is a fundamental domain for a natural group action on the cone of big divisors. The group is generated by reflections, which presumably correspond to walls where $l_{0,\pm}$ induce divisorial contractions; we expect them to behave similarly to the "bouncing walls" appearing in Sections 8 and 9. The two maps $l_\pm$ of equation (2) for the two adjacent chambers can likely be identified via the monodromy operators introduced in [Mar08, Mar03].

Open questions. Theorem 1.4, and in particular case (b), does not treat the case of "totally semistable walls", which is the case where there is no σ_0-stable complex. Proving a similar result in general would lead to further progress towards determining the movable cone for moduli spaces of stable sheaves (for general results and conjectures on the movable cone for an Hyperkähler manifold, see [HT09, HT10]); in particular, it would likely imply the above-mentioned conjecture on Lagrangian fibrations for any moduli space of Gieseker-stable sheaves on a K3 surface.

We will treat this case in [BM].

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Notation and Convention. For an abelian group G and a field $k(=\mathbb{Q},\mathbb{R},\mathbb{C})$, we denote by G_k the k-vector space $G\otimes k$.

Throughout the paper, X will be a smooth projective variety over the complex numbers. For a (locally-noetherian) scheme (or algebraic space) S, we will use the notation $\mathrm{D}^{\mathrm{b}}(S)$ for its bounded derived category of coherent sheaves, $\mathrm{D}_{qc}(S)$ for the unbounded derived category of quasi-coherent sheaves, and $\mathrm{D}_{S\text{-perf}}(S\times X)$ for the category of *S-perfect complexes*. (An S-perfect complex is a complex of $\mathcal{O}_{S\times X}$-modules which locally, over S, is quasi-isomorphic to a bounded complex of coherent shaves which are flat over S.)

We will abuse notation and denote all derived functors as if they were underived. We denote by p_S and p_X the two projections from $S\times X$ to S and X, respectively. Given $\mathcal{E}\in\mathrm{D}_{qc}(S\times X)$, we denote the Fourier-Mukai functor associated to $\mathcal{E}$ by

$$\Phi_{\mathcal{E}}(\_) := (p_X)_*\left(\mathcal{E}\otimes p_S^*(\_)\right).$$

We let $K_{\mathrm{num}}(X)$ be the numerical Grothendieck group of X and denote by $\chi(-)$ (resp., $\chi(-,-)$) the Euler characteristic on $K_{\mathrm{num}}(X)$: for $E,F\in\mathrm{D}^{\mathrm{b}}(X)$,

$$\chi(E)=\sum_p(-1)^p\,h^p(X,E)$$
$$\chi(E,F)=\sum_p(-1)^p\,\mathrm{ext}^p(E,F).$$

We denote by $\mathrm{NS}(X)$ the Néron-Severi group of X, and write $N^1(X):=\mathrm{NS}(X)_{\mathbb{R}}$. The space of full numerical stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ will be denoted by $\mathrm{Stab}(X)$.

Given a complex $E\in\mathrm{D}^{\mathrm{b}}(X)$, we denote its cohomology sheaves by $\mathcal{H}^*(E)$. The skyscraper sheaf at a point $x\in X$ is denoted by $k(x)$. For a complex number $z\in\mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively.

For a K3 surface X, we denote the Mukai vector of an object $E \in \mathrm{D^b}(X)$ by $v(E)$. We will often write it as $v(E) = (r, c, s)$, where r is the rank of E, $c \in \mathrm{NS}(X)$ and s the degree of $v(E)$. For a spherical object S we denote the spherical twist at S by $\mathrm{ST}_S(\_)$, defined in [ST01] by the exact triangle, for all $E \in \mathrm{D^b}(X)$,

$$\mathrm{Hom}^\bullet(S, E) \otimes S \to E \to \mathrm{ST}_S(E).$$

2. Review: Bridgeland stability conditions

In this section, we give a brief review of stability conditions on derived categories, as introduced in [Bri07].

Let X be a smooth projective variety, and denote by $\mathrm{D^b}(X)$ its bounded derived category of coherent sheaves. A *full numerical stability condition* σ on $\mathrm{D^b}(X)$ consists of a pair $(Z, \mathcal{A})$, where $Z\colon K_{\mathrm{num}}(X) \to \mathbb{C}$ is a group homomorphism (called *central charge*) and $\mathcal{A} \subset \mathrm{D^b}(X)$ is the *heart of a bounded t-structure*, satisfying the following three properties:

(a) For any $0 \neq E \in \mathcal{A}$ the central charge $Z(E)$ lies in the following semi-closed upper half-plane:

$$Z(E) \in \mathbb{H} := \mathcal{H} \cup \mathbb{R}_{<0} = \mathbb{R}_{>0} \cdot e^{(0,1]\cdot i\pi} \tag{3}$$

This positivity condition is the essential ingredient for our positivity result. One could think it as two separate positivity conditions: $\Im Z$ defines a rank function on the abelian category $\mathcal{A}$, i.e., a non-negative function $\mathrm{rk}\colon \mathcal{A} \to \mathbb{R}_{\geq 0}$ that is additive on short exact sequences. Similarly, $-\Re Z$ defines a degree function $\deg\colon \mathcal{A} \to \mathbb{R}$, which has the property that $\mathrm{rk}(E) = 0 \Rightarrow \deg(E) > 0$. We can use them to define a notion of slope-stability with respect to Z on the abelian category $\mathcal{A}$ via the slope $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$.

(b) With this notion of slope-stability, every object in $E \in \mathcal{A}$ has a Harder-Narasimhan filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n = E$ such that E_i/E_{i-1} are Z-semistable, with $\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$.

(c) There is a constant $C > 0$ such that, for all Z-semistable object $E \in \mathcal{A}$, we have

$$\|E\| \leq C|Z(E)|,$$

where $\|*\|$ is a fixed norm on $K_{\mathrm{num}}(X) \otimes \mathbb{R}$.

The last condition was called *support property* in [KS08], and is equivalent (see [BM11, Proposition B.4]) to Bridgeland's notion of a *full* stability condition.

Definition 2.1. A stability condition is called *algebraic* if its central charge takes value in $\mathbb{Q} \oplus \mathbb{Q}\sqrt{-1}$.

As $K_{\mathrm{num}}(X)$ is finitely generated, this implies that the image of Z is a discrete lattice in $\mathbb{C}$.

Given $(Z, \mathcal{A})$ as above, one can extend the notion of stability to $\mathrm{D}^{\mathrm{b}}(X)$ as follows: for $\phi \in (0,1]$, we let $\mathcal{P}(\phi) \subset \mathcal{A}$ be the full subcategory Z-semistable objects with $Z(E) \in \mathbb{R}_{>0}e^{i\phi}$; for general ϕ, it is defined by the compatibility $\mathcal{P}(\phi+n) = \mathcal{P}(\phi)[n]$. Each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian. Its nonzero objects are called *σ-semistable* of phase ϕ, and its simple objects are called *σ-stable*. Then each object $E \in \mathrm{D}^{\mathrm{b}}(X)$ has a *Harder-Narasimhan filtration*, where the inclusions $E_{i-1} \subset E_i$ are replaced by exact triangles $E_{i-1} \to E_i \to A_i$, and where A_i are σ-semistable of decreasing phases ϕ_i. The category $\mathcal{P}(\phi)$ necessarily has finite length. Hence every object in $\mathcal{P}(\phi)$ has a finite Jordan-Hölder filtration, whose filtration quotients are σ-stable objects of the phase ϕ. Two objects $A, B \in \mathcal{P}(\phi)$ are called *S-equivalent* if their Jordan-Hölder factors are the same (up to reordering).

The set of stability conditions will be denoted by $\mathrm{Stab}(X)$. It has a natural metric topology (see [Bri07, Prop. 8.1] for the explicit form of the metric). Bridgeland's main theorem is the following:

Theorem 2.2 (Bridgeland)**.** *The map*

$$\mathcal{Z}\colon \mathrm{Stab}(X) \to \mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{C}), \qquad (Z, \mathcal{A}) \mapsto Z,$$

is a local homeomorphism. In particular, $\mathrm{Stab}(X)$ *is a complex manifold of finite dimension equal to the rank of* $K_{\mathrm{num}}(X)$.

In other words, a stability condition $(Z, \mathcal{A})$ can be deformed uniquely given a deformation of its central charge Z.

Let us now fix a class $v \in K_{\mathrm{num}}(X)$, and consider the set of σ-semistable objects $E \in \mathrm{D}^{\mathrm{b}}(X)$ of class v as σ varies. The proof of the following statement is essentially contained in [Bri08, Section 9]; see also [BM11, Proposition 3.3] and [Tod08, Prop 2.8]:

Proposition 2.3. *There exists a locally finite set of* walls *(real codimension one submanifolds with boundary) in* $\mathrm{Stab}(X)$*, depending only on* v*, with the following properties:*

(a) When σ varies within a chamber, the sets of σ-semistable and σ-stable objects of class v does not change.
(b) When σ lies on a single wall $W \subset \mathrm{Stab}(X)$, then there is a σ-semistable object that is unstable in one of the adjacent chambers, and semistable in the other adjacent chamber.
(c) When we restrict to an intersection of finitely many walls $W_1, \dots, W_k$, we obtain a wall-and-chamber decomposition on $W_1 \cap \dots \cap W_k$ with the same properties, where the walls are given by the intersections $W \cap W_1 \cap \dots \cap W_k$ for any of the walls $W \subset \mathrm{Stab}(X)$ with respect to v.

If v is primitive, then σ lies on a wall if and only if there exists a strictly σ-semistable object of class v. The Jordan-Hölder filtration of σ-semistable objects does not change when σ varies within a chamber.

Definition 2.4. Let $v \in K_{\mathrm{num}}(X)$. A stability condition is called *generic* with respect to v if it does not lie on a wall in the sense of Proposition 2.3.

We will also need the following observation, which is a small variant of [Tod08, Lemma 2.9]:

Lemma 2.5. *Consider a stability condition $\sigma = (Z, \mathcal{A})$ with $Z(v) = -1$. Then there are algebraic stability conditions $\sigma_i = (Z_i, \mathcal{A}_i)$ for $i = 1, \dots, m$ nearby σ with $Z_i(v) = -1$ such that:*

(a) For each i, an object of class v is σ_i-stable or σ_i-semistable if and only if it is σ-stable or σ-semistable, respectively.
(b) The central charge Z is in the convex hull of $\{Z_1, \dots, Z_n\}$.

Proof. If v is generic, this follows immediately from Theorem 2.2 and Proposition 2.3, and the density of algebraic central charges $\mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{Q} \oplus i\mathbb{Q})$ inside the vector space $\mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{C})$. Once we restrict to the subset $Z(v) = -1$, any wall is locally defined by a linear rational equation of the form $\Im Z(w) = 0$, where $w \in K_{\mathrm{num}}(X)$ is the class of a destabilizing subobject, and thus the claim follows similarly. ☐

Remark 2.6. There are two group actions on $\mathrm{Stab}(X)$, see [Bri07, Lemma 8.2]: the group of autoequivalences $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ acts on the left via $\Phi(Z, \mathcal{A}) = (Z \circ \Phi_*^{-1}, \Phi(\mathcal{A}))$, where $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ and Φ_* is the automorphism induced by Φ at the level of numerical Grothendieck groups. The universal cover $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ of the group $\mathrm{GL}_2^+(\mathbb{R})$ of matrices with positive determinant acts on the right as a lift of the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{C}) \cong \mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{R}^2)$.

3. Positivity

In this section we prove our main result, Positivity Lemma 3.3.

We consider any smooth projective variety X with a numerical stability condition $\sigma = (Z, \mathcal{A})$ on $\mathrm{D}^{\mathrm{b}}(X)$. Let us first recall the definition of flat families, due to Bridgeland:

Definition 3.1. Let $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}(X)$ be the heart of a bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$. Let S be an algebraic space of finite type over $\mathbb{C}$, and let $\mathcal{E} \in \mathrm{D}_{S\text{-perf}}(S \times X)$. We say that $\mathcal{E}$ is *flat* with respect to $\mathcal{A}$ if, for every closed point $s \in S$, the derived restriction $\mathcal{E}_s$ belongs $\mathcal{A}$.

Let $v \in K_{\mathrm{num}}(X)$ be a class with $Z(v) = -1$, and denote by $\mathfrak{M}$ be the moduli stack of flat families of σ-semistable objects of class v and phase 1 (we will make this notion precise in Section 5, in the case of K3 surfaces). Our construction in this section gives a version of Theorem 1.2 for the stack $\mathfrak{M}$; we will discuss how it extends to the coarse moduli space (when it exists) in Section 4.

Proposition and Definition 3.2. *To any projective curve C with a map $C \to \mathfrak{M}$ we associate a number $\ell_\sigma.C$ as follows: let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(C \times X)$ be the corresponding universal family, and let $\Phi_{\mathcal{E}} \colon \mathrm{D}^{\mathrm{b}}(C) \to \mathrm{D}^{\mathrm{b}}(X)$ be the associated Fourier-Mukai transform. Then*

$$\ell_\sigma.C := \Im Z(\Phi_{\mathcal{E}}(\mathcal{O}_C)). \tag{4}$$

This has the following properties:

(a) Modifying the universal family by tensoring with a pull-back of a line bundle on C does not modify $\ell_\sigma.C$.
(b) We can replace $\mathcal{O}_C$ in equation (4) *by any line bundle on C, without changing $\ell_\sigma.C$.*

We will think of ℓ_σ as a divisor class in $N^1(\mathfrak{M})$.

Proof. If $c \in C$, then replacing the universal family by $\mathcal{E}' = \mathcal{E} \otimes p^*\mathcal{O}_C(c)$ effects the Fourier-Mukai transform by

$$[\Phi_{\mathcal{E}'}(\mathcal{O}_C)] = [\Phi_{\mathcal{E}}(\mathcal{O}_C)] + [\Phi_{\mathcal{E}}(k(c))] = [\Phi_{\mathcal{E}}(\mathcal{O}_C)] + v.$$

As $\Im Z(v) = 0$, this proves claim (a), and similarly (b). $\square$

Positivity Lemma 3.3. *The divisor class ℓ_σ is nef: $\ell_\sigma.C \geq 0$. Further, we have $\ell_\sigma.C > 0$ if and only if for two general closed points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'} \in \mathrm{D}^{\mathrm{b}}(X)$ are not S-equivalent.*

We first point out that by Lemma 2.5, we can immediately restrict to the case where σ is an algebraic stability condition. This implies that the heart $\mathcal{A}$ is Noetherian, by [AP06, Proposition 5.0.1].

The essential ingredient in the proof is the construction and description by Abramovich and Polishchuk of a constant family of t-structures on $S \times X$ induced by $\mathcal{A}$, for smooth S given in [AP06] and extended to singular S in [Pol07]. For any scheme S of finite type over $\mathbb{C}$, we denote by $\mathcal{A}_S$ the heart of the "constant t-structure" on $\mathrm{D}^{\mathrm{b}}(S \times X)$ given by [Pol07, Theorem 3.3.6]. The heart $\mathcal{A}_S$ could be thought of as $\mathrm{Coh}\, S \boxtimes \mathcal{A}$, since it behaves like $\mathcal{A}$ with respect to X, and like $\mathrm{Coh}\, S$ with respect to S. For example, whenever $F \in \mathrm{Coh}\, S$ and $E \in \mathcal{A}$, we have $F \boxtimes E \in \mathcal{A}_S$; also, $\mathcal{A}_S$ is invariant under tensoring with line bundles pulled back from S. It is characterized by the following statements (which paraphrase [Pol07, Theorem 3.3.6]):

Theorem 3.4. *Let $\mathcal{A}$ be the heart of a Noetherian bounded t-structure on $\mathrm{D}^{\mathrm{b}}(X)$. Denote by $\mathcal{A}^{qc} \subset \mathrm{D}_{qc}(X)$ the closure of $\mathcal{A}$ under infinite coproducts in the derived category of quasi-coherent sheaves.*

(a) For any scheme S of finite type of $\mathbb{C}$ there is a Noetherian bounded t-structure on $\mathrm{D}^{\mathrm{b}}(S \times X)$, whose heart $\mathcal{A}_S$ is characterized by

$$\mathcal{E} \in \mathcal{A}_S \Leftrightarrow p_*\mathcal{E}|_{X \times U} \in \mathcal{A}^{qc} \quad \textit{for every open affine } U \subset S$$

(b) *The above construction defines a sheaf of t-structures over S: when $S = \bigcup_i U_i$ is an open covering of S, then $\mathcal{E} \in \mathcal{A}_S$ if and only if $\mathcal{E}|_{X \times U_i} \in \mathcal{A}_{U_i}$ for every i.*

(c) *When S is projective and $\mathcal{O}_S(1)$ denotes an ample divisor, then*

$$\mathcal{E} \in \mathcal{A}_S \Leftrightarrow (p_X)_*(\mathcal{E} \otimes p_S^*\mathcal{O}_S(n)) \in \mathcal{A} \quad \textit{for all } n \gg 0.$$

The following lemma is essentially [Pol07, Proposition 2.3.7] (see also [AP06, Corollary 3.3.3] for the smooth case):

Lemma 3.5. *Let $\mathcal{E} \in \mathrm{D^b}(S \times X)$ be a flat family of objects in $\mathcal{A}$. Then $\mathcal{E} \in \mathcal{A}_S$.*

Proof. We first claim the statement when S is a zero-dimensional scheme of finite length $l > 0$, with a unique closed point $s \in S$. Choose a filtration

$$0 = F_0 \subset F_1 \subset \ldots \subset F_l = \mathcal{O}_S,$$

of the structure sheaf in $\mathrm{Coh}\, S$ with $F_i/F_{i-1} \cong k(s)$ for all i. After pull-back to $S \times X$ and tensoring with $\mathcal{E}$ we get a sequence of morphisms in $\mathrm{D^b}(S \times X)$

$$0 = G_0 \to G_1 \to \ldots \to G_l = \mathcal{E},$$

such that $\mathrm{cone}(G_{i-1} \to G_i) \cong \mathcal{E}_s$ for all i. By induction on i we obtain $(p_X)_*G_i \in \mathcal{A}$; then part (c) of Theorem 3.4 implies $\mathcal{E} \in \mathcal{A}_S$.

For general S, any closed point $s \in S$ is contained in a local zero-dimensional subscheme $T \subset S$ that is a local complete intersection in S. The previous case shows $\mathcal{E}_T \in \mathcal{A}_T$, and by [Pol07, Proposition 2.3.7] we can cover S by open sets U with $\mathcal{E}_U \in \mathcal{A}_U$. By the sheaf property (b), this shows $\mathcal{E} \in \mathcal{A}_S$. □

Lemma 3.6. *Let C be an integral projective curve, and $\mathcal{E} \in D^b(C \times X)$ be a family of σ-semistable objects in $\mathcal{P}(1)$. Then, there exists $n_0 > 0$ such that*

$$\Phi_{\mathcal{E}}(L) \in \mathcal{A},$$

for all line bundles L on C with degree $\deg(L) \geq n_0$.

Proof. By the previous lemma, we have $\mathcal{E} \in \mathcal{A}_C$. Fix an ample divisor $\mathcal{O}_C(1)$ on C. By the statement (c) of Theorem 3.4, there exists $m_0 > 0$ such that

$$\Phi_{\mathcal{E}}(\mathcal{O}_C(n)) = (p_X)_*(\mathcal{E} \otimes p_C^*\mathcal{O}_C(n)) \in \mathcal{A},$$

for all $n \geq m_0$. Fix $n_0 > 0$ such that, for a line bundle L with $\deg(L) \geq n_0$, we have $H^0(C, L(-m_0)) \neq 0$. Then, consider the exact sequence

$$0 \to \mathcal{O}(m_0) \to L \to T \to 0,$$

where T has zero-dimensional support. By applying $\Phi_{\mathcal{E}}$ we get our claim. □

Lemma 3.6 directly implies the first claim of Positivity Lemma 3.3: For a curve $C \to \mathfrak{M}$ with universal family $\mathcal{E} \in \mathrm{D^b}(X \times C)$, we have

$$\ell_\sigma.C = \Im Z\left(\Phi_{\mathcal{E}}(\mathcal{O}_C)\right) = \Im Z\left(\Phi_{\mathcal{E}}(\mathcal{O}_C(n))\right) \geq 0,$$

by the basic positivity property of the central charge Z in equation (3).

It remains to prove the second claim.

Lemma 3.7. *Let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S \times X)$ be a flat family of semistable objects in $\mathcal{A}$ over an irreducible scheme S of finite type over $\mathbb{C}$. Assume that the union of all Jordan-Hölder factors of $\mathcal{E}_s$ over all closed points $s \in S$ is finite. Then all the objects $\mathcal{E}_s$ are S-equivalent to each other, and we can choose a Jordan-Hölder filtration for every $\mathcal{E}_s$ such that the order of their stable filtration quotients does not depend on s.*

Proof. If we choose a Jordan-Hölder filtration of $\mathcal{E}_s$ for every closed point s, then there will be a stable object $F \in \mathcal{A}$ that appears as the final quotient $\mathcal{E}_s \twoheadrightarrow F$ of the filtration for infinitely many $s \in S$. In particular, $\mathrm{Hom}(\mathcal{E}_s, F)$ is non-zero for infinitely many $s \in S$; by semi-continuity, this implies that for *every* $s \in S$ there is a (necessarily surjective) morphism $\mathcal{E}_s \twoheadrightarrow F$ in $\mathcal{A}$. The same argument applied to the kernel of $\mathcal{E}_s \twoheadrightarrow F$ implies the claim by induction on the length of $\mathcal{E}_{s_0}$ for a fixed chosen point $s_0 \in S$. $\square$

Lemma 3.8. *Let $F \in \mathcal{A}$ be a simple object. Then any subobject of $p_X^* F$ in $\mathcal{A}_S$ is of the form $I \boxtimes F$ for some ideal sheaf $I \subset \mathcal{O}_S$ on S.*

Proof. By the sheaf property of $\mathcal{A}_S$, it is sufficient to treat the case where S is affine. By the characterization (a) in Theorem 3.4 of $\mathcal{A}_S$, a subobject $G \subset p_X^* F$ in $\mathcal{A}_S$ gives a subobject $(p_X)_* G$ of $(p_X)_* p_X^* F = \mathcal{O}_S \otimes_{\mathbb{C}} F$ in $\mathcal{A}^{qc}$ that is compatible with the $\mathcal{O}_S$-module structure (see also [Pol07, Proposition 3.3.7]). Since F is simple, such a subobject must be of the form $I \otimes_{\mathbb{C}} F$ for some ideal $I \subset \mathcal{O}_S$. $\square$

Lemma 3.9. *Let $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(C \times X)$ be a family of semistable objects over an irreducible reduced curve C. Assume that for general $c, c' \in C$, the objects $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent to each other. Then there exist line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on C and a filtration*

$$0 = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \mathcal{E}$$

in $\mathcal{A}_S$ such that, for all $i = 1, \dots, n$,

$$\Gamma_i / \Gamma_{i-1} \cong F_i \boxtimes \mathcal{L}_i$$

and such the restrictions of the Γ_i to the fibers $\{c\} \times X$ induces the Jordan-Hölder filtration of $\mathcal{E}_c$ for all but finitely many $c \in C$.

Proof. The same arguments as in the previous Lemma show that any two $\mathcal{E}_c, \mathcal{E}_{c'}$ are S-equivalent to each other, and that there is a common stable subobject $F := F_1 \subset \mathcal{E}_s$ for all s.

We first claim that there exists a line bundle $\mathcal{L}_1$ on C with a non-zero morphism $\phi \colon F \boxtimes \mathcal{L}_1 \to \mathcal{E}$ on $C \times X$; equivalently, we need to show that for $\mathcal{L} := \mathcal{L}_1^*$, we have

$$0 \neq \mathrm{Hom}_X \left(F, (p_X)_* (\mathcal{E} \otimes p_C^* \mathcal{L}) \right) = \mathrm{Hom}_X \left(F, \Phi_{\mathcal{E}}(\mathcal{L}) \right).$$

Let n_0 be as in Lemma 3.6, and fix a line bundle $\mathcal{L}_0$ on C of degree n_0. Set $r := \dim \operatorname{Ext}^1(F, \Phi_{\mathcal{E}}(\mathcal{L}_0))$. Pick $r+1$ distinct smooth points $c_1, \dots, c_{r+1} \in C$, and set $\mathcal{L} := \mathcal{L}_0(c_1 + \dots + c_{r+1})$. Consider the short exact sequence

$$0 \to \mathcal{L}_0 \to \mathcal{L} \to \mathcal{O}_{c_1} \oplus \dots \oplus \mathcal{O}_{c_{r+1}} \to 0$$

in $\operatorname{Coh} C$. By Lemma 3.6, it induces a short sequence

$$0 \to \Phi_{\mathcal{E}}(\mathcal{L}_0) \to \Phi_{\mathcal{E}}(\mathcal{L}) \to \mathcal{E}_{c_1} \oplus \dots \oplus \mathcal{E}_{c_{r+1}} \to 0$$

in $\mathcal{A}$. Since $\dim \operatorname{Hom}(F, \mathcal{E}_{c_1} \oplus \dots \oplus \mathcal{E}_{c_{r+1}}) \geq r+1 > r = \dim \operatorname{Ext}^1(F, \Phi_{\mathcal{E}}(\mathcal{L}_0))$, there exists a non-zero morphism from F to the direct sum that factors via $\Phi_{\mathcal{E}}(\mathcal{L})$, which proves the existence of the morphism ϕ as claimed.

It follows from [Lie06, Proposition 2.2.3] that the restriction $\phi_c \colon F_1 \to \mathcal{E}_c$ is non-zero for all but finitely many closed points $c \in C$. Since F_1 is stable, this morphism is necessarily injective. To proceed by induction, it remains to show that ϕ is an injective morphism in $\mathcal{A}_C$. Otherwise, by Lemma 3.8 the kernel of ϕ is of the form $I \otimes \mathcal{L}_1 \boxtimes F_1$ for some ideal sheaf $I \subset \mathcal{O}_C$. In particular, the derived restriction of $\ker \phi \hookrightarrow \mathcal{L}_1 \boxtimes F_1$ to $\{c\} \times X$ is an isomorphism for all but finitely many c, in contradiction to the injectivity of ϕ_c. □

Proof. (Positivity Lemma 3.3) Let C be an integral projective curve, and let $\mathcal{E} \in \mathfrak{M}(C)$. As we observed before, we only need to prove the second claim.

"$\Leftarrow$": Assume that $\ell_\sigma.C = 0$. We will show that all objects $\mathcal{E}_c$, for smooth points $c \in C$, are S-equivalent to each other.

By Lemma 3.6, for a line bundle L of large degree on C, and for $c \in C$ a smooth point, the short exact sequence

$$0 \to L(-c) \to L \to k(c) \to 0$$

induces a short exact sequence in $\mathcal{A}$

$$0 \to \Phi_{\mathcal{E}}(L(-c)) \to \Phi_{\mathcal{E}}(L) \to \mathcal{E}_c \to 0.$$

Since $l_\sigma.C = 0$, we have $Z(\Phi_{\mathcal{E}}(L)) \in \mathbb{R}_{<0}$, and so $\Phi_{\mathcal{E}}(L) \in \mathcal{P}(1)$ is semistable, of the same phase as $\mathcal{E}_c$. It follows that the Jordan-Hölder factors of $\mathcal{E}_c$ are a subset of the Jordan-Hölder factors of $\Phi_{\mathcal{E}}(\mathcal{L})$, which of course do not depend on c. Lemma 3.7 implies the claim.

"$\Rightarrow$": Assume that, for general $c \in C$, all objects $\mathcal{E}_c$ are S-equivalent to each other. Using Lemma 3.9 and the projection formula, we obtain:

$$\begin{aligned} \ell_\sigma.C = \Im Z\left([\Phi_{\mathcal{E}}(\mathcal{O}_C)]\right) &= \sum_{i=1}^n \Im Z\left([(p_X)_* F_i \boxtimes \mathcal{L}_i]\right) \\ &= \sum_{i=1}^n \Im Z\left([F_i \otimes H^\bullet(C, \mathcal{L}_i)]\right) = \sum_{i=1}^n \chi(C, \mathcal{L}_i) \cdot \Im Z\left([F_i]\right) = 0 \end{aligned}$$

□

4. A NATURAL NEF DIVISOR CLASS ON THE MODULI SPACE, AND COMPARISON

Let S be a proper algebraic space of finite type over $\mathbb{C}$, let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$, and let $\mathcal{E} \in \mathrm{D}_{S\text{-perf}}(S \times X)$ be a flat family of semistable objects of class v with $Z(v) = -1$. By restriction of the family, our construction assigns a number $l_\sigma.C$ to every curve $C \subset S$.

Theorem 4.1. *The assignment $C \mapsto l_\sigma.C$ only depends on the numerical curve class $[C]$, and is additive on curve classes. It defines a nef divisor $\ell_\sigma \in N^1(S)$, which is invariant under replacing the universal family with the tensor product of a line bundle pulled back from S. Additionally, for a curve $C \subseteq S$ we have $\ell_\sigma.C > 0$ if and only if for two general closed points $c, c' \in C$, the corresponding objects $\mathcal{E}_c, \mathcal{E}_{c'} \in \mathrm{D}^{\mathrm{b}}(X)$ are not S-equivalent.*

Proof. If C_1, C_2 are numerically equivalent, then $[\mathcal{O}_{C_1}], [\mathcal{O}_{C_2}] \in K_{\mathrm{num}}(S)$ only differ by multiples of the class of a skyscraper sheaf $k(s)$ of a closed point s. Since $\Phi_{\mathcal{E}}$ preserves numerical equivalence, and since $\Im Z(\Phi_{\mathcal{E}}(k(s))) = \Im Z(v) = 0$, this proves the first claim. The additivity follows similarly, and all other claims follow directly from the Positivity Lemma 3.3. □

Example 4.2. Let $X \subset \mathbb{P}^4$ be a smooth quintic threefold, containing two disjoint lines L_1, L_2. Consider the smooth proper algebraic space X^+ obtained as the flop of X at the line L_1. Then, by a classical result of Bondal and Orlov [BO95], we have an equivalence of derived categories $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}(X^+)$. However, X^+ admits no numerically positive class, as the flopped curve $\widetilde{L}_1$ is the negative of L_2 in $N_1(X^+)$. By Theorem 1.2, X^+ cannot be isomorphic to a moduli space of (semi)stable complexes on $\mathrm{D}^{\mathrm{b}}(X)$ with respect to a numerical stability condition on X.

There are many examples of non-projective flops of a projective variety, including Mukai flops of holomorphic symplectic varieties. By the same reasoning, these non-projective flops cannot be obtained by wall-crossing.

We will now compare our construction to the classical notion of a determinant divisor on S associated to the family $\mathcal{E}$ (see, e.g., [Muk87, Don90, LP92, Fal93, Li96] and [HL10, Section 8.1]).

Definition 4.3. We define a group homomorphism (called the *Donaldson morphism*)

$$\lambda_{\mathcal{E}} : v^\sharp \to N^1(S)$$

as the composition

$$v^\sharp \xrightarrow{p_X^*} K_{\mathrm{num}}(S \times X)_{\mathbb{R}} \xrightarrow{\cdot[\mathcal{E}]} K_{\mathrm{num}}(S \times X)_{\mathbb{R}} \xrightarrow{(p_S)_*} K_{\mathrm{num}}(S)_{\mathbb{R}} \xrightarrow{\det} N^1(S),$$

where

$$v^\sharp := \{ w \in K_{\mathrm{num}}(X)_{\mathbb{R}} \, : \, \chi(v \cdot w) = 0 \} .$$

Since the Euler characteristic χ gives a non-degenerate pairing, we can write[3]

$$\Im(Z(\_)) = \chi(w_Z \cdot \_),$$

for a unique vector $w_Z \in v^\sharp$.

Proposition 4.4. *For an integral curve $C \subset S$, we have*

$$\lambda_{\mathcal{E}}(w_Z).C = \Im Z(\Phi_{\mathcal{E}}(\mathcal{O}_C)) =: \ell_{\sigma,\mathcal{E}}.C. \tag{5}$$

Proof. It is enough to prove (5) when $w_Z = [F] \in K_{\mathrm{num}}(X)$, for some $F \in \mathrm{D}^{\mathrm{b}}(X)$. We define

$$\mathcal{L}(F) := (p_S)_* \left(p_X^* F \otimes \mathcal{E}\right).$$

As in the classical case, the assumption $w_Z \in v^\sharp$ and the projection formula show that the rank of $\mathcal{L}(F)$ must be equal to 0:

$$\mathrm{rk}\,\mathcal{L}(F) = \chi(S, \mathcal{O}_s \otimes \mathcal{L}(F)) = \chi(S \times X, \mathcal{O}_{\{s\}\times X} \otimes \mathcal{E} \otimes p_X^* F) = \chi(\mathcal{E}_s \otimes F) = \chi(v \cdot w_Z) = 0.$$

The Riemann-Roch Theorem gives

$$\lambda_{\mathcal{E}}(w_Z).C = \deg \mathcal{L}(F)|_C = \chi(C, \mathcal{L}(F)|_C).$$

By Cohomology and Base Change (see, e.g., [Kuz06, Corollary 2.23]) we deduce that

$$\mathcal{L}(F)|_C = (p_C)_* \left(p_X^* F \otimes \mathcal{E}|_C\right).$$

Using the projection formula again gives

$$\chi(C, \mathcal{L}(F)|_C) = \chi(S, F \otimes (p_X)_* \mathcal{E}|_C) = \chi(w_Z \cdot \Phi_{\mathcal{E}}(\mathcal{O}_C)) = \Im Z(\Phi_{\mathcal{E}}(\mathcal{O}_C)).$$

□

The basic properties of $\ell_{\sigma,\mathcal{E}}$ are in [HL10, Lemma 8.1.2]. In particular, we recall the following: Let $\mathcal{N}$ be a vector bundle on S of rank n. Then

$$\ell_{\sigma,\mathcal{E}\otimes p_S^*\mathcal{N}} = n \cdot \ell_{\sigma,\mathcal{E}}. \tag{6}$$

By Theorem 4.1, we have a well-defined positive divisor class on a fine moduli space of stable complexes. To extend this to the case when a universal family exists only étale locally on a coarse moduli space, we recall the following definition from [Muk87].

Definition 4.5. Let S be an algebraic space of finite-type over $\mathbb{C}$.

(a) A flat family $\mathcal{E}$ on $S \times X$ is called a *quasi-family* of objects in $\mathfrak{M}$ if, for all closed points $t \in T$, there exist an integer $\rho > 0$ and an element $E \in \mathfrak{M}$ such that $\mathcal{E}|_{\{t\}\times X} \cong E^{\oplus \rho}$. If T is connected, the positive integer ρ does not depend on t and it is called the *similitude* of $\mathcal{E}$.

[3] When we will consider the case of K3 surfaces, we will change notation and denote by $v^\perp$, in place of $v^\sharp$, the orthogonal with respect to the Mukai pairing, and, accordingly, with w_Z the vector representing $\Im(Z(\_))$ with respect to the Mukai pairing. The only difference is a plus/minus sign in the rank and the degree component of the vector. Finally, the corresponding Donaldson morphism will be denoted θ_v.

(b) Two quasi-families $\mathcal{E}$ and $\mathcal{E}'$ on $T \times X$ are called *equivalent* if there exist vector bundles V and V' on T such that $\mathcal{E}' \otimes p_T^* V \cong \mathcal{E} \otimes p_T^* V'$.
(c) A quasi-family $\mathcal{E}$ is called *quasi-universal* if, for every scheme T' and for any quasi-family $\mathcal{T}$ on $T \times X$, there exists a unique morphism $f \colon T' \to T$ such that $f^*\mathcal{E}$ and $\mathcal{T}$ are equivalent.

By [Muk87, Theorem A.5], if $\mathfrak{M}$ consists only of stable complexes, and if $\mathfrak{M}$ is a $\mathbb{C}^*$-gerbe over an algebraic space M of finite-type over $\mathbb{C}$ (i.e., over its coarse moduli space), then there exists a quasi-universal family on $M \times X$.

Let $\mathcal{E}$ be a quasi-family in $\mathfrak{M}(T)$ of similitude ρ, for T a proper algebraic space. We can define a divisor class ℓ_σ in T (and a Donaldson morphism $\lambda_{\mathcal{E}}$) by $l_\sigma := \frac{1}{\rho} \cdot l_{\sigma,\mathcal{E}}$ that does not depend on the choice of quasi-universal family, by (6). We obtain therefore the analogue version of Theorem 4.1 for quasi-universal families; in particular, it has the same positivity properties.

Example 4.6. Let X be a smooth projective K3 surface. We will see in Lemma 5.6 that, when $\mathfrak{M}$ consists only of stable objects, we have a quasi-universal family on the proper algebraic space M, and therefore a divisor class, denoted $\ell_\sigma(v)$. The class $\ell_\sigma(v)$ depends only on σ, it is strictly positive on all curves, and it is compatible with $\ell_{\sigma,\mathcal{E}}$ via pull-back.

Example 4.7. Let X, Y be a smooth projective varieties, where Y is equipped with a Brauer class $\alpha \in \mathrm{Br}(Y)$ and a polarization $H \in \mathrm{NS}(Y)$. Denote by $\mathrm{D^b}(Y, \alpha)$ the derived category of α-twisted sheaves on Y. Assume that there is a derived equivalence $\Phi\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y, \alpha)$ such that

(a) H is a *generic polarization* with respect to $v' := \Phi(v)$, and
(b) $\mathfrak{M}_{\Phi(\sigma)}(v')$ consists of twisted H-Gieseker semistable sheaves on Y.

Then a coarse moduli space M exists and ℓ_σ defines a divisor class on it, which depends only on σ, with the positive property as in Theorem 1.2, and compatible with $\ell_{\sigma,\mathcal{E}}$ via pull-back. When $\Phi = \mathrm{id}$, this is proved in [HL10, Theorem 8.1.5] and [LP05, Théorème 5 & Proposition 6]. When Φ is non-trivial, then the result follows from [Orl97, CS07]: the equivalence is of Fourier-Mukai type, and the construction of ℓ_σ is compatible with the convolution of the Fourier-Mukai kernels.

We refer to [O'G97] for the notion of a generic polarization; it always exists when v' has positive rank. Some references for twisted sheaves and stability are [Căl00, HS05, Yos06, Lie07].

5. REVIEW: STABILITY CONDITIONS AND MODULI SPACES FOR OBJECTS ON K3 SURFACES

In this section we give a brief review of Bridgeland's results on stability conditions for K3 surfaces in [Bri08], and of results by Toda, Yoshioka and others related to moduli spaces of Bridgeland-stable objects.

Space of stability conditions for a K3 surface. Let X be a smooth projective K3 surface. Let $v\colon K_{\mathrm{num}}(X) \to H^*_{\mathrm{alg}}(X,\mathbb{Z})$ be the Mukai vector given by $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$. We denote the Mukai pairing $H^*_{\mathrm{alg}}(X,\mathbb{Z}) \times H^*_{\mathrm{alg}}(X,\mathbb{Z}) \to \mathbb{Z}$ by $(\_,\_)$; it can be defined by $(v(E), v(F)) := -\chi(E,F)$. Given a Mukai vector $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$, we denote its orthogonal complement by

$$v^\perp := \left\{ w \in H^*_{\mathrm{alg}}(X,\mathbb{R}) \,:\, (v,w) = 0 \right\}.$$

Fix $\omega, \beta \in \mathrm{NS}(X)_{\mathbb{Q}}$ with ω ample. We define a slope function $\mu_{\omega,\beta}$ on $\mathrm{Coh}\, X$ by

$$\mu_{\omega,\beta}(\mathcal{E}) = \begin{cases} \frac{\omega.(c_1(\mathcal{E})-\beta)}{r(\mathcal{E})} & \text{if } r(\mathcal{E}) > 0, \\ +\infty & \text{if } r(\mathcal{E}) = 0. \end{cases} \tag{7}$$

This gives a notion of slope-stability, for which Harder-Narasimhan filtrations exist. Let $\mathcal{T}(\omega,\beta) \subset \mathrm{Coh}\, X$ be the subcategory of sheaves whose HN-filtrations factors have $\mu_{\omega,\beta} > 0$, and $\mathcal{F}(\omega,\beta)$ the subcategory of sheaves with HN-filtration factors satisfying $\mu_{\omega,\beta} \leq 0$. Next, consider the abelian category

$$\mathcal{A}(\omega,\beta) := \left\{ E \in \mathrm{D}^{\mathrm{b}}(X) : \begin{array}{l} \bullet\ \mathcal{H}^p(E) = 0 \text{ for } p \notin \{-1,0\}, \\ \bullet\ \mathcal{H}^{-1}(E) \in \mathcal{F}(\omega,\beta), \\ \bullet\ \mathcal{H}^0(E) \in \mathcal{T}(\omega,\beta) \end{array} \right\} \tag{8}$$

and the $\mathbb{C}$-linear map

$$Z_{\omega,\beta}\colon K_{\mathrm{num}}(X) \to \mathbb{C}, \qquad E \mapsto (\exp(\beta + \sqrt{-1}\omega), v(E)). \tag{9}$$

If $Z_{\omega,\beta}(F) \notin \mathbb{R}_{\leq 0}$ for all spherical sheaves $F \in \mathrm{Coh}(X)$ (e.g., this holds when $\omega^2 > 2$), then by [Bri08, Lemma 6.2, Prop. 7.1], the pair $\sigma_{\omega,\beta} = (Z_{\omega,\beta}, \mathcal{A}(\omega,\beta))$ defines a stability condition. For objects $E \in \mathcal{A}(\omega,\beta)$, we will denote their phase with respect to $\sigma_{\omega,\beta}$ by $\phi_{\omega,\beta}(E) = \phi(Z(E)) \in (0,1]$.

Denote by $U(X) \subset \mathrm{Stab}(X)$ the open subset consisting of the stability conditions $\sigma_{\omega,\beta}$ just constructed up to the action of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$. It can also be characterized as the open subset $U(X) \subset \mathrm{Stab}(X)$ consisting of stability conditions for which the skyscraper sheaves $k(x)$ of points are stable of the same phase. Let $\mathrm{Stab}^\dagger(X) \subset \mathrm{Stab}(X)$ be the connected component containing $U(X)$. Let $\mathcal{P}(X) \subset H^*_{\mathrm{alg}}(X)_{\mathbb{C}}$ be the subset consisting of vectors whose real and imaginary parts span positive definite two-planes in $H^*_{\mathrm{alg}}(X)_{\mathbb{R}}$ with respect to the Mukai pairing. It has two connected components. Choose $\mathcal{P}^+(X) \subset \mathcal{P}(X)$ as the connected component containing the vector $(1, i\omega, -\omega^2/2)$, for $\omega \in \mathrm{NS}(X)_{\mathbb{R}}$ the class of an ample divisor. Furthermore, set $\Delta(X) := \{\delta \in H^*_{\mathrm{alg}}(X,\mathbb{Z}) \,:\, \delta^2 = -2\}$ and, for $\delta \in \Delta$,

$$\delta^\perp_{\mathbb{C}} := \left\{ \Omega \in H^*_{\mathrm{alg}}(X,\mathbb{C}) :\ (\Omega,\delta) = 0 \right\}.$$

Finally, set

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta_{\mathbb{C}}^{\perp} \subset K_{\mathrm{num}}(X)_{\mathbb{C}}.$$

Since the Mukai pairing $(\_, \_)$ is non-degenerate, we can define $\eta(\sigma) \in H^*_{\mathrm{alg}}(X)_{\mathbb{C}}$ for a stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}^{\dagger}(X)$ by

$$\mathcal{Z}(\sigma)(\_) = (\_, \eta(\sigma)).$$

Theorem 5.1 (Bridgeland)**.** *The map $\eta\colon \mathrm{Stab}^{\dagger}(X) \to H^*_{\mathrm{alg}}(X)_{\mathbb{C}}$ is a covering map onto its image $\mathcal{P}_0^+(X)$.*

The proof of Theorem 5.1 relies on an explicit description of the boundary $\partial U(X)$ of $U(X)$, see [Bri08, Theorem 12.1]:

Theorem 5.2. *Suppose that $\sigma = (Z, \mathcal{P}) \in \partial U(X)$ is a general point of the boundary of $U(X)$. Then exactly one of the following possibilities holds.*

(A^+) *There is a rank r spherical vector bundle A such that the only stable factors of the objects $\{k(x) : x \in X\}$ in the stability condition σ are A and $\mathrm{ST}_A(k(x))$. Thus, the Jordan-Hölder filtration of each $k(x)$ is given by*

$$0 \to A^{\oplus r} \to k(x) \to \mathrm{ST}_A(k(x)) \to 0.$$

(A^-) *There is a rank r spherical vector bundle A such that the only stable factors of the objects $\{k(x) : x \in X\}$ in the stability condition σ are $A[2]$ and $\mathrm{ST}_A^{-1}(k(x))$. Thus, the Jordan-Hölder filtration of each $k(x)$ is given by*

$$0 \to \mathrm{ST}_A^{-1}(k(x)) \to k(x) \to A^{\oplus r}[2] \to 0.$$

(C_k) *There are a nonsingular rational curve $C \subset X$ and an integer k such that $k(x)$ is stable in the stability condition σ, for $x \notin C$, and such that the Jordan-Hölder filtration of $k(x)$, for $x \in C$, is given by*

$$0 \to \mathcal{O}_C(k+1) \to k(x) \to \mathcal{O}_C(k)[1] \to 0.$$

In the previous theorem, by a *general point of the boundary* we mean a stability condition which lies on only one wall (in the sense of Proposition 2.3). By ST_A we denote the spherical twist functor of [ST01] associated to the spherical object A.

Remark 5.3. In the boundary of type (C_k), the Mukai vectors of the stable factors of $k(x)$ span a negative definite plane in $H^*_{\mathrm{alg}}(X)_{\mathbb{R}}$.

Remark 5.4. The results stated in this section extend without any difference to the case of twisted K3 surfaces. Let (X, α) be a twisted K3 surface, $\alpha \in \mathrm{Br}(X)$. Stability conditions on (X, α) were studied in [HMS08, Section 3.1]. In particular, we can introduce the following analogue objects to the untwisted case: $\mathrm{NS}(X, \alpha)$, $\mathrm{Stab}^{\dagger}(X, \alpha)$, $U(X, \alpha)$, $\mathcal{P}_0^+(X, \alpha)$. Theorems 5.1 and 5.2 hold with a similar statement in the twisted setting as well.

Moduli stacks of semistable objects. Let $\mathrm{Sch}_{\mathbb{C}}$ be the site of schemes locally of finite type over $\mathbb{C}$, endowed with the étale topology. Define a 2-functor, with values in the category Grp of groupoids,

$$\mathfrak{G}\colon \mathrm{Sch}_{\mathbb{C}} \to \mathrm{Grp},$$

by mapping a $\mathbb{C}$-scheme S to the groupoid $\mathfrak{G}(S)$, whose objects consists of those $\mathcal{E} \in \mathrm{D}_{S\text{-perf}}(S \times X)$ which satisfy $\mathrm{Ext}^i(\mathcal{E}_s, \mathcal{E}_s) = 0$, for all $i < 0$ and all closed points $s \in S$, and whose morphisms are isomorphisms in $\mathrm{D}_{S\text{-perf}}(S \times X)$. Here $\mathcal{E}_s$ is the derived restriction of $\mathcal{E}$ to $\mathrm{D}^{\mathrm{b}}(\{s\} \times X) \cong \mathrm{D}^{\mathrm{b}}(X)$. The main theorem in [Lie06] (generalizing results in [Ina02]) shows that $\mathfrak{G}$ is an Artin stack, locally of finite type over $\mathbb{C}$.

Fix $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$, $\phi \in \mathbb{R}$, and $v \in H^*_{\mathrm{alg}}(X)$. We define a sub 2-functor $\mathfrak{M}_\sigma(v, \phi) \subset \mathfrak{G}$ of "flat families of σ-semistable objects of class v and phase ϕ": its objects are given by objects $\mathcal{E} \in \mathrm{D}_{S\text{-perf}}(S \times X)$ whose restrictions $\mathcal{E}_s$ belong to $\mathcal{P}(\phi)$ and have Mukai vector v, for all closed points $s \in S$. We will often omit ϕ from the notation.

Theorem 5.5 ([Tod08, Theorem 1.4 and Section 3])**.** *Let X be a K3 surface and let $\sigma \in \mathrm{Stab}^\dagger(X)$. Then σ-stability is an open property, i.e., $\mathfrak{M}_\sigma(v, \phi) \subset \mathfrak{G}$ is an open substack, and $\mathfrak{M}_\sigma(v, \phi)$ is an Artin stack of finite type over $\mathbb{C}$.*

Let $\mathfrak{M}^s_\sigma(v, \phi) \subseteq \mathfrak{M}_\sigma(v, \phi)$ be the open substack parameterizing stable objects. Inaba proved in [Ina02] that $\mathfrak{M}^s_\sigma(v, \phi)$ is a $\mathbb{G}_m$-gerbe over a symplectic algebraic space $M^s_\sigma(v, \phi)$.

Lemma 5.6. *Fix $\phi \in \mathbb{R}$ and $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$.*

(a) The moduli stack $\mathfrak{M}_\sigma(v, \phi)$ satisfies the valuative criterion of universal closedness.
(b) Assume that $\mathfrak{M}_\sigma(v, \phi) = \mathfrak{M}^s_\sigma(v, \phi)$. Then the coarse moduli space $M_\sigma(v, \phi)$ is a proper algebraic space.

Proof. Using the $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$, we may assume that $\phi = 1$ and $Z(v) = -1$. By Lemma 2.5, we may assume that σ is algebraic, and hence $\mathcal{A}$ is Noetherian. In this case, [AP06, Theorem 4.1.1] implies the Lemma. $\square$

Moduli spaces of semistable objects. Let $v = (r, c, s) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive class. Let us first consider moduli spaces of Gieseker stable sheaves. We say that v is *positive* (as in [Yos01b, Definition 0.1]) if

- $v^2 \geq -2$,
- $r \geq 0$, and
- if $r = 0$, then c must be effective and $s \neq 0$.

By [Yos01b, Theorems 0.1 & 8.1] (see also [KLS06, Section 2.4], where the condition of positivity is discussed more in detail), if v is primitive and positive, then for a general ample line bundle H on X, the moduli space of H-Gieseker semistable sheaves $M_H(v)$

on X with Mukai vector v is non-empty, consists of stable sheaves, and it is a smooth projective irreducible symplectic manifold of dimension v^2+2.

On the other hand, fix $\beta \in \mathrm{NS}(X)$ with $\mu_{H,\beta}(v) > 0$. Then [Bri08, Proposition 14.1] shows that for $\omega = tH$ and $t \gg 0$, being $\sigma_{\omega,\beta}$-stable of class v is equivalent to being (β-twisted) Gieseker stable, and so $M_{\sigma_{\omega,\beta}}(v) = M_H(v)$.

Theorem 5.7 (Toda, Yoshioka)**.** *Let $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$. Assume that $v = mv_0$, with $m \in \mathbb{Z}_{>0}$ and v_0 a primitive vector with $v_0^2 \geq -2$. Then $\mathfrak{M}_\sigma(v,\phi)(\mathbb{C})$ is non-empty for all $\sigma = (Z,\mathcal{A}) \in \mathrm{Stab}^\dagger(X)$ and all $\phi \in \mathbb{R}$ with $Z(v) \in \mathbb{R}_{>0} \cdot e^{i\phi\pi}$.*

Proof. Since we are interested in semistable objects, we can assume that $v = v_0$ is primitive. Also, since being semistable is a closed condition on $\mathrm{Stab}(X)$, we can assume that σ is generic with respect to v, so that every σ-semistable object of class v is stable. Then the Joyce invariant $J(v)$ of [Tod08] is the motivic invariant of the projective coarse moduli space $M_\sigma(v)$.

By [Tod08, Theorem 1.4], $J(v)$ does not depend on σ, and it is invariant under autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$. Hence, up to acting by the shift functor, tensoring with a line bundle, and the spherical twist $\S\,\mathrm{ST}_{\mathcal{O}}$, we can assume that v is positive (as defined above), and that $J(v)$ is equal to the motivic invariant of the moduli space $M_H(v)$ of Gieseker stable sheaves on X with Mukai vector v, for a generic polarization H. Since v is positive, Yoshioka's result shows that $M_H(v)$ is non-empty. Hence, $J(v)$ is non-trivial, and so $\mathfrak{M}_\sigma(w,\phi)(\mathbb{C})$ is non-empty for all σ. □

By Theorem 5.7 and [Ina11], we get the following corollary.

Corollary 5.8. *Let $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$ be a primitive vector with $v^2 \geq -2$, and let $\sigma \in \mathrm{Stab}^\dagger(X)$ be a generic stability condition with respect to v. Then $M_\sigma(v)$ is non-empty, consists of stable objects, and it is a smooth proper symplectic algebraic space of dimension v^2+2.*

Let us briefly recall the Beauville-Bogomolov form on $\mathrm{NS}(M)$ for a symplectic algebraic space M. It is a bilinear form $\mathrm{NS}(M) \times \mathrm{NS}(M) \to \mathbb{R}$. If $\omega \in H^0(M,\Omega_M^2)$ is a global non-degenerate two-form, then there is a constant c such that

$$(D_1,D_2) = c\int_M D_1 D_2 (\omega\bar{\omega})^{\frac{1}{2}\dim M - 1}.$$

Its associated quadratic form is denoted by $q(D) = (D,D)$. It determines the volume of D by $\int_M D^{\dim M} = q(D)^{\frac{1}{2}\dim M}$. In particular, a divisor D is big if and only if $q(D) > 0$ (see also [Huy99, Proposition 6.5]).

As mentioned in the footnote to Definition 4.3, in the case of a K3 surface we will always consider a dual version $\theta_v \colon v^\perp \to N^1(M_\sigma(v)$ of the Donaldson morphism, defined by $\theta_v(w) = -\lambda_{\mathcal{E}}(w^*)$. This morphism is also called the *Mukai homomorphism*, and is

compatible with the Mukai pairing on X in the following sense: if $C \subset M_\sigma(v)$ is a curve, and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(M_\sigma(v))$ a universal family, then

$$\theta_v(w).C = (w, \Phi_{\mathcal{E}}(\mathcal{O}_C)). \tag{10}$$

The following result is proved in [Yos01b, Sections 7 & 8]:

Theorem 5.9 (Yoshioka). *Let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive vector with $v^2 \geq -2$. Let $\sigma \in \mathrm{Stab}^\dagger(X)$ be a generic stability condition with respect to v. Assume that there exist a K3 surface Y, a Brauer class $\alpha \in \mathrm{Br}(Y)$, a polarization $H \in \mathrm{NS}(Y)$, and a derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y, \alpha)$ such that*

(a) $v' = \Phi(v)$ is positive,
(b) H is generic with respect to v, and
(c) $M_{\Phi(\sigma)}(v')$ consists of twisted H-Gieseker stable sheaves on Y.

Then the Donaldson morphism induces an isomorphism

- $\theta_v\colon v^\perp \xrightarrow{\sim} \mathrm{NS}(M_\sigma(v))$, *if* $v^2 > 0$*;*
- $\theta_v\colon v^\perp/v \xrightarrow{\sim} \mathrm{NS}(M_\sigma(v))$, *if* $v^2 = 0$.

Under this isomorphism, the quadratic Beauville-Bogomolov form for $\mathrm{NS}(M_\sigma(v))$ coincides with the quadratic form of the Mukai pairing on X.

Proof. When we deal with Gieseker stable sheaves, then this is nothing but [Yos01b, Theorem 8.1] (see also [KLS06, Section 2.4]). The general case follows from [Orl97, CS07], as in Example 4.7.

The last statement is proved by deformation to the Hilbert scheme; see [Yos01b] for the case of abelian surfaces, and also [GNY09, Section 1.5]. □

6. K3 SURFACES: PROJECTIVITY OF MODULI SPACES

Let X be a smooth projective K3 surface, and let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$. In the recent preprint [MYY11b], Minamide, Yanagida, and Yoshioka proved the following: if $\mathrm{NS}(X) \cong \mathbb{Z}$ and $\sigma \in \mathrm{Stab}^\dagger(X)$ is a generic stability condition with respect to v, then there exist another K3 surface Y, a Brauer class $\alpha \in \mathrm{Br}(Y)$, and a derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y, \alpha)$ such that the moduli stack $\mathfrak{M}_\sigma(v)$ is isomorphic to a moduli stack of (twisted) Gieseker semistable sheaves on (Y, α) via Φ.

In this section, we improve their argument, and we remove the assumption on the rank of the Néron-Severi group. As a consequence, the divisor class ℓ_σ gives an ample divisor on the coarse moduli space.

We write $v = mv_0 \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, where $m \in \mathbb{Z}_{>0}$, and $v_0 = (r, c, s)$ is primitive with $v_0^2 \geq -2$. We start by examining the cases in which $v_0^2 \leq 0$.

Lemma 6.1. *Assume that $v_0^2 = -2$. Then, for all $\sigma \in \mathrm{Stab}^\dagger(X)$ generic with respect to v, $\mathfrak{M}_\sigma(v)$ admits a coarse moduli space consisting of a single point. In particular, it is non-empty.*

Proof. Corollary 5.8 shows that for all generic $\sigma \in \mathrm{Stab}^\dagger(X)$, the stack $\mathfrak{M}_\sigma(v_0) = \mathfrak{M}^s_\sigma(v_0) \neq \emptyset$ is a $\mathbb{G}_m$-gerbe over a point. The corresponding object E_0 is spherical, and in particular admits no non-trivial self-extensions. If $m > 1$, then $v^2 < -2$ shows that there cannot exist any stable object with vector v. By induction, every semistable object with Mukai vector v must be of the form $E_0^{\oplus m}$. □

Lemma 6.2. *Assume that $v_0^2 = 0$. Let $\sigma \in \mathrm{Stab}^\dagger(X)$ be a generic stability condition with respect to v. Then, we have:*

(a) for $m = 1$, $\mathfrak{M}_\sigma(v_0) = \mathfrak{M}^s_\sigma(v_0) \neq \emptyset$, $M_\sigma(v_0)$ is a smooth projective K3 surface, and there exists a class $\alpha \in \mathrm{Br}(M_\sigma(v_0))$ such that the choice of a universal family (étale locally) induces a derived equivalence

$$\Phi_{\sigma,v_0} \colon \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(M_\sigma(v_0), \alpha).$$

(b) for $m > 1$, $\mathfrak{M}_\sigma(v) \neq \emptyset$, a coarse moduli space $M_\sigma(v)$ exists and

$$M_\sigma(v) \cong \mathrm{Sym}^m\left(M_\sigma(v_0)\right).$$

Proof. Corollary 5.8 shows the non-emptiness. The fact that $M_\sigma(v_0)$ is a smooth projective K3 surface and the derived equivalence is a classical result of Mukai [Muk87] for stable sheaves, whose generalization to stable complexes follows by standard techniques. This shows (a).

The proof of (b) follows now as in [MYY11b]. Indeed, clearly $\mathfrak{M}_\sigma(v) \neq \emptyset$, and the derived equivalence Φ maps any complex in $\mathfrak{M}_\sigma(v)(\mathbb{C})$ in a torsion sheaf on $M_\sigma(v_0)$ of dimension 0 and length m. □

We can now prove Theorem 1.3, based on an idea of Minamide, Yanagida, and Yoshioka. By Lemma 6.1 and Lemma 6.2, we can restrict to the case $v^2 > 0$.

The following result is proved in [MYY11b, Sections 4.1 and 3.4] for abelian surfaces, and for K3 surfaces of Picard rank one. For convenience of the reader, we give a self-contained proof for arbitrary K3 surfaces:

Lemma 6.3. *Let $\sigma = (Z_\sigma, \mathcal{A}_\sigma)$ be a generic stability condition with respect to v, lying inside a chamber $\mathcal{C}$ with respect to v. Then $\mathcal{C}$ contains a dense subset of stability condition $\tau = (Z_\tau, \mathcal{A}_\tau)$ for which there exists primitive Mukai vector w with $w^2 = 0$ such that:*

(a) $Z_\tau(w)$ and $Z_\tau(v)$ lie on the same ray in the complex plane.

(b) All τ-semistable objects with Mukai vector w are stable, and $M_\tau(w)$ is a smooth projective K3 surface.

Proof. Let us consider claim (a). We may assume $Z_\sigma(v) = -1$ and restrict our attention to stability conditions τ with $Z_\tau(v) = -1$. Let $Q \subset H^*_{\mathrm{alg}}(X)_\mathbb{R}$ be the quadric defined by $w^2 = 0$. Due to the signature of the Mukai pairing, there is a real solution w_r to the pair of equations $\Im Z_\sigma(w) = 0$ and $w^2 = 0$. Since Q has a rational point, rational points are dense in Q, i.e., there exists $w_q \in H^*_{\mathrm{alg}}(X)_\mathbb{Q}$ arbitrarily close to w_r with $w_q^2 = 0$.

If w_q is sufficiently close, and since w_q must be linearly independent of v, there will be $\tau = (Z_\tau, \mathcal{A}_\tau)$ nearby σ such that $\Im Z_\tau(v) = \Im Z_\tau(w_q) = 0$ and $\Re Z_\tau = \Re Z_\sigma$. Replacing w_q by the unique primitive integral class $w \in \mathbb{R} \cdot w_q$ with $\Re Z_\tau(w) < 0$ finishes the proof of the first claim.

It remains to show that claim (b) holds, after possibly replacing w and a further deformation of τ.

Note that small deformations of τ in a codimension one submanifold of $\mathrm{Stab}(X)$ will keep property (a) intact. If this contains a stability conditions generic with respect to w, our claim follows from Lemma 6.2. Otherwise, we can assume that τ is on a generic point of a wall, and that for $u \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, the complex number $Z(u)$ has the same phase as $Z(v)$ and $Z(w)$ if and only if u is a linear combination of v and w.

Using the Fourier-Mukai transform associated to $M_\rho(w)$ for ρ nearby τ and generic, we can further assume that $w = (0, 0, 1)$ is the Mukai vector of a point, and that ρ is in a generic boundary point of the geometric chamber $U(X)$ as described in Theorem 5.2. If $M_\rho(v)$ is not a fine moduli space, we need to consider $M_\rho(v)$ as a twisted K3 surface; see Remark 5.4.

In the case of a wall of type (A^+), let w' be the Mukai vector of $\mathrm{ST}_A(k(x))$. Since the objects $\mathrm{ST}_A(k(x))$ are τ-stable, the stability condition τ is generic with respect to w', we have $w'^2 = 0$, and $Z(w')$ has the same phase as $Z(v)$. The case (A^-) is analogous.

If we are in case (C_k), then as pointed out in Remark 5.3, the Mukai pairing is negative definite on the linear span $\langle w, v(\mathcal{O}_C(k+1))\rangle$. However, since $Z(\mathcal{O}_C(k+1))$ has the same phase as $Z(v)$, this linear span is equal to the linear span $\langle v, w\rangle$ of v, w, in contradiction to $v^2 > 0$. □

Let w be the Mukai vector from Lemma 6.3. Let $Y := M_{\sigma_{\omega,\beta}}(w)$, and let $\alpha \in \mathrm{Br}(Y)$ be a Brauer class so that the choice of a universal family induces a derived equivalence $\Phi : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y, \alpha)$. Consider the stability condition $\sigma' := \Phi(\sigma_{\omega,\beta}) \in \mathrm{Stab}(Y, \alpha)$. Then, by construction, for all $F \in \mathfrak{M}_{\sigma_{\omega,\beta}}(w)(\mathbb{C})$, $\Phi(F) \cong k(y)$, for some $y \in Y$. Hence, if w does not lie in a wall for v, $k(y)$ is not a stable factor (with respect to σ') for $\Phi(E)$, for all $E \in \mathfrak{M}_{\sigma_{\omega,\beta}}(v)(\mathbb{C})$. By [Bri08, Lemma 10.1], $\Phi(E)[-1]$ is a α-twisted locally-free sheaf on Y, which is μ_H-semistable, for some polarization H on Y. Again, since w is not a wall for v, all sheaves $\Phi(E)[-1]$ are twisted H-Gieseker stable, and the polarization H is generic. This shows that $\Phi \circ [-1]$ induces an isomorphism of stacks

$$\mathfrak{M}_{\sigma_{\omega,\beta}}(v) \xrightarrow{\sim} \mathfrak{M}_H(\Phi(v)), \tag{11}$$

where $\mathfrak{M}_H(\Phi(v))$ denotes the moduli stack of twisted H-Gieseker semistable sheaves on Y. This shows that a coarse moduli space for $\mathfrak{M}_{\sigma_{\omega,\beta}}(v)$ exists and it is a normal irreducible projective variety with $\mathbb{Q}$-factorial singularities (by [KLS06, PR11]).

Finally, if $\sigma \in \mathrm{Stab}^\dagger(X)$, then as in [Tod08, Section 4.3], we can reduce, up to derived autoequivalences of $\mathrm{D^b}(X)$, to the case $\sigma = \sigma_{\omega,\beta} \in U(X)$. This concludes the proof of the first part of Theorem 1.3.

We can now show the second part of Theorem 1.3, namely that ℓ_σ is well-defined on the coarse moduli space $M_\sigma(v)$ and it is ample.

Lemma 6.4. *Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}^\dagger(X)$ be such that $Z(v) = -1$, and let $w_\sigma := \Im(\eta(\sigma))$. Then $w_\sigma^2 > 0$.*

Proof. This follows directly from Theorem 5.1, since $\eta(\sigma) \in \mathcal{P}_0^+(X)$. □

First of all, when v is primitive and σ generic with respect to v, by Example 4.6, the divisor class ℓ_σ is well-defined on the coarse moduli space $M_\sigma(v)$.

Lemma 6.5. *Let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive vector with $v^2 \geq 2$, and $\sigma \in \mathrm{Stab}^\dagger(X)$ be generic with respect to v. Then for all $u \in v^\perp$ we have $q(\theta_v(u)) = u^2$, where $q(\theta_v(u))$ denotes the quadratic Beauville-Bogomolov form on $M_\sigma(v)$.*

Proof. By the first part of Theorem 1.3, we can reduce to the case where $M_\sigma(v)$ is a moduli space of Gieseker stable locally-free sheaves; this case is treated in Theorem 5.9. □

Corollary 6.6. *With the same assumptions as in Lemma 6.5, the divisor ℓ_σ is ample.*

Proof. By the Lemma, $q(\ell_\sigma) = w_\sigma^2$. By Lemma 6.4, $w_\sigma^2 > 0$, and so ℓ_σ is big. As a symplectic algebraic space, $M_\sigma(v)$ has trivial canonical bundle; so the Base Point Free Theorem [KM98, Theorem 3.3] implies that ℓ_σ is globally generated, and hence ample (see also [Huy99, Proposition 6.5]). □

The case in which v is not primitive is more delicate, since we do not have a version of Lemma 6.5 available. Instead, we have to use an explicit comparison with determinant line bundles and rely on the GIT construction for dealing with properly semistable objects; we use [HL10, Section 8.1] as reference for the classical construction.

By the openness and convexity of the ample cone, it is sufficient to prove the ampleness of ℓ_σ for a dense subset of stability conditions in a given chamber. Let us first assume that σ satisfies the properties of the stability condition τ in Lemma 6.3; let Φ be the induced derived equivalence $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y, \alpha)$. We will first assume $\alpha = 0$. By Example 4.7, we know that ℓ_σ gives a well-defined class on the GIT quotient producing the coarse moduli space $M_\sigma(v)$. Since skyscraper sheaves $k(y)$ of points on Y are stable with respect to $\Phi(\sigma)$, we have $\Phi(\sigma) \in U(Y)$. Since the $\Phi(\sigma)$-stable objects $\Phi(E)$ for $E \in M_\sigma(v)$ have the same phase as skyscraper sheaves $k(y)$, we can use the $\widetilde{\mathrm{GL}}^+$-action to normalize the stability condition $\Phi(\sigma)$ such that its central charge W is of the form $W = (e^{i\omega+\beta}, v)$ given in equation (9), and satisfies $W(\Phi_*(v)) \in \mathbb{R}_{<0}$ at the same time. If we write $-\Phi_*(v) = (r, c, s)$, this is equivalent to

$$\omega.c - r\omega.\beta = 0. \tag{12}$$

Note that by the construction of Lemma 6.3, the stability condition $\Phi_*(\sigma)$ is still generic with respect to $\Phi_*(v)$.

We first claim that ω-slope (semi)stability for sheaves of class $-\Phi_*(v)$ is equivalent to twisted ω-Gieseker (semi)stability: indeed, assume that a sheaf $\mathcal{E}$ with $v(\mathcal{E}) = -\Phi_*(v)$ is slope-semistable. Recall that $\mathcal{P}_{\omega,\beta}(1)$ is the category generated by skyscrapher sheaves $k(x)$, and by shifts $\mathcal{F}[1]$ of ω-slope semistable sheaves $\mathcal{F}$ satisfying (12). In particular, if $\mathcal{F} \subset \mathcal{E}$ is a saturated subsheaf of the same slope, then $\mathcal{F}[1]$ is a subobject of $\mathcal{E}[1]$ in $\mathcal{P}_{\omega,\beta}(1)$; since $\Phi_*(\sigma)$ is generic with respect to $\Phi_*(v)$, this means that $v(\mathcal{F})$ is proportional to $v(\mathcal{E})$; hence the twisted Hilbert polynomial of $\mathcal{F}$ is proportional to the twisted Hilbert polynomial of $\mathcal{E}$, and this will hold independently of the twist β. In particular, twisted Gieseker stability on Y for $-\Phi_*(v)$ is equivalent to untwisted Gieseker-stability.

In other words, in the isomorphism (11) we can in fact take $\mathfrak{M}_H(\Phi(v))$ to be the classical moduli stack of Gieseker semistable sheaves. Let $\mathcal{L}_0, \mathcal{L}_1$ be as defined in [HL10, Definition 8.1.9]; after identifying h of [HL10, Section 8.1] with ω, then in our notation we have $\mathcal{L}_0 = \theta_v((-r, 0, s))$ and $\mathcal{L}_1 = \theta_v((0, r\omega, \omega.c))$. It is immediate to check that, up to rescaling, ℓ_σ coincides with the class $\mathcal{L}_1$. By Theorem 1.2, ℓ_σ is nef. By [HL10, Theorem 8.1.11 & Remark 8.1.12], we have that $\mathcal{L}_0 \otimes \mathcal{L}_1^{\otimes m}$ is ample for $m \gg 0$. Moreover, $\mathcal{L}_0 \otimes \mathcal{L}_1^{\otimes m}$ for $m \gg 0$ are (up to rescaling) induced by a stability conditions arbitrarily close to σ. Hence we have found a dense subset of stability conditions for which ℓ_σ is ample.

Finally, in case $\alpha \neq 0$, one can use Proposition 2.3.3.6 and Lemma 2.3.2.8 of [Lie07] to reduce to the case $\alpha = 0$. This finishes the proof of Theorem 1.3.

7. Flops via wall-crossing

In this section, we will first discuss the possible phenomena at walls in $\mathrm{Stab}(X)$, and then proceed to prove Theorem 1.4 .

Let X be a smooth projective K3 surface, let v be a primitive Mukai vector with $v^2 \geq -2$. Consider a wall $W \subset \mathrm{Stab}(X)$ with respect to v in the sense of Proposition 2.3.

Let $\sigma_0 \in W$ be a generic point on the wall. Let $\sigma_+ = (Z_+, \mathcal{A}_+), \sigma_- = (Z_-, \mathcal{A}_-)$ be two algebraic stability conditions in the two adjacent chambers. By the results of the previous section, the two moduli spaces $M_\pm := M_{\sigma_\pm}(v)$ are non-empty, irreducible symplectic projective manifolds. If we choose (quasi-)universal families $\mathcal{E}_\pm$ on $M_\pm$ of $\sigma_\pm$-stable objects, we obtain in particular (quasi-)families of σ_0-semistable objects. Hence, Theorem 4.1 gives us nef divisor classes $\ell_{0,\pm} := \ell_{\sigma_0,\mathcal{E}_\pm}$ on $M_\pm$.

There are several possible phenomena at the wall, depending on the codimension of the locus of strictly σ_0-semistable objects, and depending on whether there are curves $C \subset M_\pm$ of S-equivalent objects with respect to σ_0, i.e., curves with $\ell_{0,\pm}.C = 0$. We call the wall W

(a) a *fake wall* there are no curves in $M_\pm$ of objects that are S-equivalent to each other with respect to σ_0,
(b) a *totally semistable wall*, if $M^s_{\sigma_0}(v) = \emptyset$,

(c) a *flopping wall*, if W is not a fake wall and $M^s_{\sigma_0}(v) \subset M_\pm$ has complement of codimension at least two,
(d) a *bouncing wall*, if there is an isomorphism $M_+ \cong M_-$ that maps $l_{0,+}$ to $l_{0,-}$, and there are divisors $D_\pm \subset M_\pm$ that are covered by curves of objects that are S-equivalent to each other with respect to σ_0.

Note that a wall can be both fake and totally semistable. In the case of a fake wall, W does not get mapped to a wall of the nef cone. In the case of a bouncing wall, the map $l_+ \colon \mathcal{C}_+ \to N^1(M_+)$ sends W to a boundary of the nef cone of $M_+ = M_-$; and so does l_-. Hence the image of a path crossing the wall W under $l_\pm$ will bounce back into the ample cone once it hits the boundary of the nef cone in N^1. We will see examples of every type of wall in Section 9.

We should point out that the behavior at fake walls and bouncing walls can exhibit different behaviors than the possibilities observed in [CI04] in different context: in general, the two universal families over M_+, M_- do not seem to be related via a derived autoequivalence.

We can assume that σ_0 is algebraic, $W_0(v) = -1$, and $\phi = 1$. By Lemma 6.5 and Lemma 6.4, $\ell_{0,\pm}$ has positive self-intersection. Since both $M_\pm$ have trivial canonical bundles, we can apply the Base Point Free Theorem [KM98, Theorem 3.3], which shows that $\ell_{0,\pm}$ are both semi-ample.

We denote the induced contraction morphism (cf. [Laz04, Theorem 2.1.27]) by

$$\pi_{\sigma_\pm} \colon M_{\sigma_\pm}(v) \to Y_\pm,$$

where $Y_\pm$ are normal irreducible projective varieties. We denote the induced ample divisor class on $Y_\pm$ by ℓ_0. If $M^s_{\sigma_0}(v) \neq \emptyset$, we denote by $f_{\sigma_0} \colon M_+(v) \dashrightarrow M_-(v)$ the induced birational map.

Note that $\pi_{\sigma_\pm}$ is an isomorphism if and only if the wall W is a fake wall, a divisorial contraction if W is a bouncing wall, and a wall contraction if W is a flopping wall.

We would like to say that $Y_+ = Y_-$, and that they are (an irreducible component of) the coarse moduli space of σ_0-semistable objects. The best statement we can prove in general is the following:

Proposition 7.1. *The space $Y_\pm$ has the following universal property: For any proper irreducible scheme $S \in \mathrm{Sch}_\mathbb{C}$, and for any family $\mathcal{E} \in \mathfrak{M}_{\sigma_0}(v)(S)$ such that there exists a closed point $s \in S$ for which $\mathcal{E}_s = \mathcal{E}|_{\{s\}\times X} \in \mathfrak{M}_{\sigma_\pm}(v)(\mathbb{C})$, there exists a finite morphism $q \colon T \to S$ and a natural morphism $f_{q^*\mathcal{E}} \colon T \to Y_\pm$.*

Proof. We prove the statement only for Y_+; the proof for Y_- is analogous. Let S be a proper scheme, and let $\mathcal{E}$ be a family as above. We can assume S is normal. By Toda's result, Theorem 5.5, there exists an open subset $S' \subseteq S$ such that $\mathcal{E}_s$ is σ_+-stable, for all $s \in S'$. By the universal property for M_+, there exists a natural morphism $f'_{\mathcal{E}} \colon S' \to M_+$. This induces a rational morphism $f_{\mathcal{E}} \colon S \dashrightarrow Y_+$.

Consider a resolution of singularities for $f_{\mathcal{E}}$,

$$\begin{tikzcd} & \widetilde{S} \arrow[dl, "c"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "f_{\mathcal{E}}"] & & Y_+. \end{tikzcd}$$

Then, the family $\widetilde{\mathcal{E}} \colon = (c \times \mathrm{id})^* \mathcal{E}$ on $\widetilde{S}$ gives rise to a divisor class $\ell_{\sigma_0, \widetilde{\mathcal{E}}}$ on $\widetilde{S}$ such that

$$\ell_{\sigma_0, \widetilde{\mathcal{E}}} = g^* \ell_0.$$

Since ℓ_0 is ample, also $\ell_{\sigma_0, \widetilde{\mathcal{E}}}$ is semi-ample. On the other hand, by Theorem 4.1, a curve $C \subseteq \widetilde{S}$ satisfies $\ell_{\sigma_0, \widetilde{\mathcal{E}}}.C = 0$ if and only if C parameterizes properly σ_0-semistable objects, generically with the same Jordan-Hölder filtration. But every curve in a fiber of c has this property. Hence, up to considering its Stein factorization, the morphism g factorizes through $f_{\mathcal{E}}$, as wanted. $\square$

If we can explicitly describe σ_0-semistable objects, Proposition 7.1 shows that Y_+ and Y_- are actually irreducible components of a coarse moduli space for $\mathfrak{M}_{\sigma_0}(v)$. We will see this in some examples in Sections 8 and 9.

Proof. (Theorem 1.4) It remains to prove assertion (b) of Theorem 1.4: in this case, $\mathfrak{M}^s_{\sigma_0}(v)$ is non-empty, and we can restrict to the case where $\ell_{0,\pm}$ is not ample. By openness of stability, all objects in $\mathfrak{M}^s_{\sigma_0}(v)(\mathbb{C})$ are stable with respect to $\sigma_\pm$. Write $M^0_\pm$ for the open subsets of $M_\pm$ consisting of those objects. By assumption, we also have $\mathrm{codim}(M^0_\pm, M_\pm) \geq 2$. (Note that since $M_\pm$ are smooth and symplectic, the two conditions $\mathrm{codim}(M_+ \setminus M^0_+, M_+) \geq 2$ and $\mathrm{codim}(M_- \setminus M^0_-, M_-) \geq 2$ are equivalent.)

Consider the birational map

$$f_{\sigma_0} \colon M_+ \dashrightarrow M_-,$$

induced by the isomorphism $M^0_+ \xrightarrow{\sim} M^0_-$.

Since $\mathrm{codim}(M^0_\pm, M_\pm) \geq 2$, and since $M_\pm$ are projective, numerical divisor classes on $M_\pm$ are determined by their intersection numbers with curves contained in $M^0_\pm$. Since we can choose (quasi-)universal families $\mathcal{E}_\pm$ on $M_\pm$ that agree on the open subset $M^0_\pm$, this implies that the maps $\ell \colon \mathcal{C}^\pm \to \mathrm{NS}(M_\pm)$ are identical, up to analytic continuation and identification of the Néron-Severi groups via f_{σ_0}; more precisely, we have the following equality in $\mathrm{NS}(M_+)$:

$$f^*_{\sigma_0} \ell_{\sigma_-} = \ell_{\sigma_+, Z_-},$$

where the RHS is given by

$$\ell_{\sigma_+, Z_-} \colon [C] \mapsto \Im \left(-\frac{Z_-(\Phi_{\mathcal{E}_+}(\mathcal{O}_C))}{Z_-(v)} \right),$$

for all curves $C \subset M_+$. Since $\ell_{0,+}$ is not ample, ℓ_{σ_+,Z_-} is big and not nef. Hence, the map f_{σ_0} does not extend to an isomorphism $M_+ \xrightarrow{\sim} M_-$ and we deduce the equality

$$f_{\sigma_0}^* \ell_{0,-} = \ell_{0,+}.$$

Hence, we deduce that $Y_+ = Y_-$, and that the following diagram commutes:

$$\begin{tikzcd} M_{\sigma_+}(v) \arrow[rr, dashed, "f_{\sigma_0}"] \arrow[dr, "\pi_+"'] & & M_{\sigma_-}(v) \arrow[dl, "\pi_-"] \\ & Y_+ = Y_- & \end{tikzcd} \tag{13}$$

□

8. Stable sheaves on K3 surfaces

In this section we discuss the three main theorems in examples for moduli space of stable sheaves on a K3 surface X. For K3 surfaces with Picard group of rank one, some of these examples can also be deduced by [MYY11b, Section 4.3].

Example 8.1. The simplest case is a primitive vector v with $v^2 = 0$. Then Lemma 6.2 and Theorem 5.2 give a complete picture on the possible wall-crossing phenomena. For a generic stability condition, the moduli space is a fixed smooth projective K3 surface Y derived equivalent to X. The possible walls are derived equivalent to the cases given in Theorem 5.2: In the cases (A^+) and (A^-), we have a totally semistable fake wall. In the case (C_k), we get a bouncing wall: the contraction induced by the wall is the divisorial contraction of rational (-2)-curves. After we cross the wall, the moduli space is still isomorphic to Y, but the universal family gets modified by applying the spherical twist at a line bundle supported on C; in $\mathrm{NS}(Y)$, this has the effect of a reflection at $[C]$.

We now give an explicit formula for the Mukai vector w_σ associated to a stability condition.

Lemma 8.2. *Let X be a smooth projective K3 surface. Let $v = (r, c, s) \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive Mukai vector with $v^2 \geq -2$, and let $\sigma_{\omega,\beta} \in U(X)$ be a generic stability condition with respect to v. Then the divisor class $\ell_{\sigma_{\omega,\beta}} \in N^1(M_{\sigma_{\omega,\beta}}(v))$ is a positive multiple of $\theta_v(w_{\sigma_{\omega,\beta}})$, where $w_{\sigma_{\omega,\beta}} = (R_{\omega,\beta}, C_{\omega,\beta}, S_{\omega,\beta})$ is given by*

$$\begin{aligned} R_{\omega,\beta} &= c.\omega - r\beta.\omega \\ C_{\omega,\beta} &= (c.\omega - r\beta.\omega)\,\beta + \left(s - c.\beta + r\,\frac{\beta^2 - \omega^2}{2}\right)\omega \\ S_{\omega,\beta} &= c.\omega\,\frac{\beta^2 - \omega^2}{2} + s\beta.\omega - (c.\beta)\cdot(\beta.\omega). \end{aligned}$$

Proof. Using the Definition of $\ell_{\sigma_{\omega,\beta}}$ in equation (1), and the compatiblity of θ_v with the Mukai pairing given in equation (10), we see that the vector is given by

$$w_{\sigma_{\omega,\beta}} = \Im \frac{e^{i\omega+\beta}}{-(e^{i\omega+\beta}, v)} \sim_{\mathbb{R}^+} -\Im\big(\overline{(e^{i\omega+\beta}, v)} \cdot e^{i\omega+\beta}\big).$$

(Here and in the following $\sim_{\mathbb{R}^+}$ will mean that the vectors are positive scalar multiples of each other.) Then the claim follows immediately from

$$e^{i\omega+\beta} = \left(1, \beta, \frac{\beta^2 - \omega^2}{2}\right) + i\,(0, \omega, \omega.\beta)\,.$$

□

If we write $\omega = t \cdot H$, for an ample divisor $H \in \mathrm{NS}(X)$, we can let t go to zero or $+\infty$. If we take the limit $t \to 0$ up to rescaling, we obtain a vector $w_{0\cdot H,\beta}$ with components

$$\begin{aligned} R_{0\cdot H,\beta} &= c.H - r\beta.H \\ C_{0\cdot H,\beta} &= (c.H - r\beta.H)\,\beta + \left(s - c.\beta + r\,\frac{\beta^2}{2}\right) H \\ S_{0\cdot H,\beta} &= c.H\,\frac{\beta^2}{2} + s\beta.H - (c.\beta)\cdot(\beta.H). \end{aligned}$$

If we similarly take the limit $t \to +\infty$, we obtain a vector $w_{\infty\cdot H,\beta}$ with components

$$\begin{aligned} R_{\infty\cdot H,\beta} &= 0 \\ C_{\infty\cdot H,\beta} &= -r\,\frac{H^2}{2} H \\ S_{\infty\cdot H,\beta} &= -c.H\,\frac{H^2}{2}. \end{aligned}$$

Example 8.3. In the previous notation, assume $v = (0, c, s)$, for v a positive primitive vector. We assume that H is a generic polarization with respect to v. Then, for t sufficiently large, $M_{\sigma_{t\cdot H,\beta}}(v) = M_H(v)$ is a Lagrangian fibration. The semi-ample nef divisor associated to this fibration is given by $w_{\infty\cdot H,\beta} \sim_{\mathbb{R}^+} (0, 0, -1)$.

The fact that $M_H(v)$ is a Lagrangian fibration can be seen by using the divisor $\theta_v(w_{\infty\cdot H,\beta})$ as follows. By Le Potier's construction, see [LP05, Section 1.3], for all $x \in X$, we can construct a section $s_x \in H^0(M_H(v), \theta_v(w_{\infty\cdot H,\beta}))$ via its zero-locus

$$V(s_x) = \{E \in M_H(v) :\ \mathrm{Hom}(E, k(x)) \neq 0\}\,.$$

The sections $\{s_x\}_{x\in X}$ generate $\theta_v(w_{\infty\cdot H,\beta})$. The induced morphism contracts the locus of sheaves with fixed support, and thus the image has lower dimension. By Matsushita's Theorem [Mat99, Mat01], the morphism is a Lagrangian fibration.

Remark 8.4. The previous example shows a general phenomenon for nef divisors obtained as an image of a wall in the space of Bridgeland stability conditions. Indeed, by Lemma 6.4 and Lemma 6.5, a divisor D coming from a wall in $\mathrm{Stab}(X)$ must have $q(D) > 0$. To obtain a nef divisor D with $q(D) = 0$ (which conjecturally corresponds to a Lagrangian fibration), we necessarily have to look at "limit points" in $\mathrm{Stab}(X)$, for example $w_{0\cdot H,\beta}$, or $w_{\infty\cdot H,\beta}$. We will use these limit points in Examples 9.5 and 9.7.

We will use the following two observations several times (for Lemma 8.5 see, e.g., [BMT11, Section 7.2]; for Lemma 8.6 see [BM11, Lemma 5.9]).

Lemma 8.5. *Let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$ be a stability condition such that*

$$\gamma := \inf\left\{\Im(Z(E)) > 0 \,:\, E \in \mathcal{A}\right\} > 0.$$

Then an object $E \in \mathcal{A}$ with $\Im(Z(E)) = \gamma$ is σ-stable if and only if $\mathrm{Hom}(\mathcal{P}(1), E) = 0$.

The previous lemma applies in particularly when $\Im(Z) \in \mathbb{Z} \cdot \gamma$, for some constant $\gamma > 0$. In this case, if an object $E \in \mathcal{A}$ with $\mathrm{Hom}(\mathcal{P}(1), E) = 0$ and $\Im(Z(E)) = 2\gamma$ is not σ-stable, then it must be destabilized by a short exact sequence $A \to E \to B$ where A and B σ-stable with $\Im(Z) = \gamma$.

Lemma 8.6. *Let $E \in \mathrm{D^b}(X)$ and $\sigma \in \mathrm{Stab}(X)$ be a stability condition such that E is σ-semistable. Assume that there is a Jordan-Hölder filtration $M^{\oplus r} \hookrightarrow E \twoheadrightarrow N$ of E such that M, N are σ-stable, $\mathrm{Hom}(E, M) = 0$, and $[E]$ and $[M]$ are linearly independent classes in $K_{\mathrm{num}}(X)$. Then σ is in the closure of the set of stability conditions where E is stable.*

Example 8.7. Let X be a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, for H an ample line bundle with $H^2 = 2d$, $d \geq 1$. Let $v = (r, cH, s)$ be a primitive Mukai vector, with $r, c, s \in \mathbb{Z}$, $r \geq 0$, $v^2 \geq -2$. We assume that there exist $A, B \in \mathbb{Z}$, $A > 0$, such that $Ac - Br = 1$.

Consider the family of stability conditions $\sigma_{t,\frac{B}{A}} := \sigma_{\omega,\beta}$ on $\mathrm{D^b}(X)$, with $\omega = t \cdot H$ and $\beta := \frac{B}{A} \cdot H$, for $t > 0$. As long as $\sigma_{t,\frac{B}{A}}$ exists, the moduli space $M_{\sigma_{t,\frac{B}{A}}}(v)$ is the moduli space of Gieseker stable sheaves $M_H(v)$: Indeed, we have

$$\Im(Z_{t,\frac{B}{A}}(\_)) \in \frac{2td}{A} \cdot \mathbb{Z},$$

and $\Im(Z_{t,\frac{B}{A}}(v)) = \frac{2td}{A}$. So Lemma 8.5 shows that Gieseker-stable sheaves are $\sigma_{t,\frac{B}{A}}$-stable.

We distinguish two cases, according to whether $\frac{dB^2+1}{A}$ is integral or not. Its relevance is explained by the fact that $w = (A, B \cdot H, \frac{dB^2+1}{A}) \in H^*_{\mathrm{alg}}(X, \mathbb{Q})$ is a class with $w^2 = -2$ and $\Im Z_{t,\frac{B}{A}}(w) = 0$; since A, B are coprime, there exists an integral class with these two properties if and only if $s\frac{dB^2+1}{A}$ is integral.

Case 1: $\frac{dB^2+1}{A} \notin \mathbb{Z}$. Then there exists no spherical object with $\Im(Z_{t,\frac{B}{A}}(v)) = 0$. By [Bri08, Proposition 7.1], all values of $t > 0$ produce a stability condition. This gives an explicit region of the ample cone of $M_H(v)$:

$$\langle \theta_v(w_{\sigma_{t,\frac{B}{A}}}) \,:\, t > 0 \rangle \subset \mathrm{Amp}(X).$$

An explicit computation is in Example 9.5.

Case 2: $\frac{dB^2+1}{A} \in \mathbb{Z}$. Then there exists a stable spherical vector bundle U satisfying $\Im(Z_{t,\frac{B}{A}}(U)) = 0$. We let $t_0 > 0$ be such that $\Re(Z_{t_0}(U)) = 0$. Then $t > t_0$ produces a line segement in the ample cone of $M_H(v)$:

$$\langle \theta_v(w_{\sigma_{t,\frac{B}{A}}}) \,:\, t > t_0 \rangle \subset \mathrm{Amp}(X).$$

The question now becomes to understand when $\mathrm{Hom}(U, F) \neq 0$, for F a Gieseker stable sheaf with Mukai vector v. An explicit computation is in the following Example.

Example 8.8. In the notation of the previous Example 8.7, we take

$$d = 1, \quad v = (2, H, s) \ \ (s \leq 0), \quad A = 1, \quad B = 0.$$

Then, the spherical vector bundle U is nothing but $\mathcal{O}_X$, and $t_0 = 1$. Up to resclaing, the vector $w_{\sigma_{t,\frac{B}{A}}}$ becomes

$$w_{\sigma_{t,0}} = \left(2t, (-2t^3 + st)H, -2t^3\right).$$

We will see that wall-crossing along this path will naturally lead to contractions of Brill-Noether loci, i.e., loci of sheaves $\mathcal{F}$ where $h^0(\mathcal{F})$ is bigger than expected. These loci and contractions have been studied in [Yos01a]. We distinguish 3 cases.

Case 1: $s = 0$. We claim that the nef cone $\mathrm{Nef}(M_H(v))$ is generated by

$$\theta_v(w_{0\cdot H,0}) \sim_{\mathbb{R}^+} \theta_v(1, 0, 0) \quad \text{and} \quad \theta_v(w_{\infty\cdot H,0}) \sim_{\mathbb{R}^+} \theta_v(0, -H, -1).$$

First of all, observe that any torsion sheaf $T \in \mathfrak{M}_H(0, H, -2)$ is a line bundle of degree -1 on a curve of genus 2; it follows that there is a short exact sequence

$$0 \to \mathcal{O}_X^{\oplus 2} \to \mathrm{ST}_{\mathcal{O}_X}^{-1}(T) \to T \to 0.$$

It easy to see that $F := \mathrm{ST}_{\mathcal{O}_X}^{-1}(T)$ is slope-stable with $v(F) = (2, H, 0)$; hence $\mathrm{ST}_{\mathcal{O}_X}^{-1}$ induces an injective morphism $M_H(0, H, -2) \to M_H(v)$, which must be an isomorphism (as they have the same dimension). Hence every $F \in \mathfrak{M}_H(v)$ is of this form, and $\mathrm{Hom}(\mathcal{O}_X, F) = \mathbb{C}^2$, for all $F \in \mathfrak{M}_H(v)(\mathbb{C})$.

To compute how the divisor class $\ell_{\sigma_{t,0}}$ varies when we cross $t = 1$, we will use Lemma 8.6. For $0 < t < 1$, we consider the stability condition $\overline{\sigma}_{t,0}$ in the boundary of $U(X)$ of type (A^+) (see Theorem 5.2). The heart $\mathcal{A}$ for $\overline{\sigma}_{t,0}$ can be explicitly described (see, e.g., [Yos09, Proposition 2.7] or [BM11, Proposition 5.6]). In particular, $\mathcal{P}(1)$ is generated by

$k(x)$ for $x \in X$, by $\mathcal{O}_X$, and by $G[1]$, where G is a μ-semistable sheaves of slope 0 satisfying $\mathrm{Hom}(\mathcal{O}_X, G) = 0$. Hence, both $\mathcal{O}_X$ and, by Lemma 8.5, any $T \in M_H(0, H, -2)$ are $\overline{\sigma}_{t,0}$-stable for all $0 < t < 1$. Similarly, the short exact sequence

$$0 \to T \to \mathrm{ST}_{\mathcal{O}_X}(T) \to \mathcal{O}_X^{\oplus 2} \to 0$$

and Lemma 8.5 show that $\mathrm{ST}_{\mathcal{O}_X}(T) = \mathrm{ST}^2_{\mathcal{O}_X}(F)$ is $\overline{\sigma}_{t,0}$-stable for all $0 < t < 1$.

In particular, $M_{\overline{\sigma}_{t,0}}(v)$ for $0 < t < 1$ is isomorphic to $M_H(v) = M_{\sigma_{t,0}}(v)$ for $1 < t$. The universal families are related by an application of $\mathrm{ST}^2_{\mathcal{O}_X}$; as this acts trivially on the K-group, the two families induces the same Donaldson morphism $v^\perp \to N^1(M_H(v))$.

To understand the wall between the two corresponding chambers, we now consider the path $\sigma_{t,-\epsilon}$, where $\epsilon > 0$ is sufficiently small such that $\mathcal{O}_X$ and all $T \in \mathfrak{M}_H(0, H, -1)$ are both $\sigma_{\frac{1}{2},-\epsilon}$-stable and $\sigma_{2,-\epsilon}$-stable. Note that the subcategory $\mathcal{A}_{t,-\epsilon}$ does not depend on t; it is then straighforward to check that $\mathcal{O}_X$ and all T are also $\sigma_{t,-\epsilon}$-stable for all $t \in [\frac{1}{2}, 2]$: indeed, the imaginary part of $Z_{t,-\epsilon}(w)$ for any Mukai vector w is of the form $t \cdot \mathrm{const}$, and the real part is of the form $\mathrm{const} + \mathrm{const} \cdot t^2$. Then the inequality $\phi_{t,-\epsilon}(w) \le \phi_{t,-\epsilon}(w')$ is equivalent to an equation of the form $\mathrm{const} \cdot t^2 \ge \mathrm{const}$.

Let $t_0 \in [\frac{1}{2}, 2]$ be such that $\mathcal{O}_X$ and $T \in \mathfrak{M}_H(0, H, -1)$ have the same phase with respect to $\sigma_{t_0,-\epsilon}$. Lemma 8.6 shows that $F = \mathrm{ST}^{-1}_{\mathcal{O}_X}(T)$ is stable for $t > t_0$, and that $\mathrm{ST}^2_{\mathcal{O}_X}(F) = \mathrm{ST}_{\mathcal{O}_X}(T)$ is stable for $t < t_0$. This is a totally semistable and fake wall.

For $t \to 0$, the contraction induced by $w_{0\cdot H,0}$ is precisely the Jacobian fibration induced by $\mathrm{ST}_{\mathcal{O}_X}$. The wall at $\beta = 1/2 \cdot H$ corresponds instead to the Uhlenbeck compactification: the corresponding divisorial contraction is induced precisely by $w_{\infty \cdot H,0}$ (see also [Lo12]).

Case 2: $s = -1$. The nef cone $\mathrm{Nef}(M_H(v))$ is generated by

$$\theta_v(w_{H,0}) = \theta_v(2, -3H, -2) \quad \text{and} \quad \theta(w_{\infty \cdot H,0}) = \theta_v(0, -H, -1).$$

Similarly to Case 1, the Riemann-Roch Theorem and stability shows $\mathrm{Hom}(\mathcal{O}_X, F) \neq 0$, for all $F \in \mathfrak{M}_H(v)(\mathbb{C})$. We can use a similar argument as before to find a wall near the singular point $\sigma_{1,0}$ where the Jordan-Hölder filtration of F is given by

$$H^0(F) \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} F \to \mathrm{cone}(\mathrm{ev}).$$

There is no stable object with Mukai vector v with respect to $\sigma_{t_0,\epsilon}$, hence we are still in the case of a totally semistable wall. Unlike in the previous case, we do have curves of S-equivalent objects that get contracted by $w_{H,0}$: there is a $\mathbb{P}^1$ parametrizing extensions

$$0 \to \mathcal{O}_X \to F \to I_\Gamma(H) \to 0, \tag{14}$$

for any zero-dimensional subscheme $\Gamma \subset X$ of length 4 contained in a curve $C \in |H|$.

Case 3: $s \le -2$. The nef cone $\mathrm{Nef}(M_H(v))$ is generated by

$$\theta_v(w_{H,0}) = \theta_v(2, (-2+s)H, -2) \quad \text{and} \quad \theta(w_{\infty \cdot H,0}) = \theta_v(0, -H, -1).$$

Indeed, in this case, we will always have both stable objects at $\sigma_{t_0,\epsilon}$ (by a dimension counting), and strictly semistable ones (corresponding to extensions as in (14), with $\Gamma \subset X$ of length $3-s$).

Finally, we proceed to give an explicit bound for the walls of the "Gieseker chamber" for any Mukai vector v, i.e., the chamber for which Bridgeland stability of objects of class v is equivalent to β-twisted Gieseker stability. In principle, this has been well-known, as all the necessary arguments are already contained in [Bri08, Proposition 14.2]; see also [Bay09, Proposition 4.1], [MYY11a, Section 2], [LQ11, Theorem 4.4]; the most explicit results can be found in [Mac12, Sections 2 and 3] (with regards to a slightly different form of the central charge) and [Kaw11]; what follows is essentially a short summary of Kawatani's argument.

We want to give a bound that is as explicit as possible for the form of the central charge given in (9). Fix a class $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$, and let ω vary on a ray in the ample cone. Given a class $v \in H^*_{\mathrm{alg}}(X)$ with positive rank and slope, Bridgeland showed that for $\omega \gg 0$, stable objects of class v are exactly the twisted-Gieseker stable sheaves; we want to give an explicit bound in terms of ω^2 and β, v that only depends on the Mukai lattice $H^*_{\mathrm{alg}}(X,\mathbb{Z})$.

Definition 8.9. Given divisor classes ω, β with ω ample, and given a class $v = (r, c, s) \in H^*_{\mathrm{alg}}(X)$ with $v^2 \geq -2$, we write $(r, c_\beta, s_\beta) = e^{-\beta}(r, c, s)$ and define its *slope* $\mu_{\omega,\beta}(v) = \frac{\omega.c_\beta}{r}$ as in Section 5, equation (7), and its *discrepancy* $\delta_{\omega,\beta}(v)$ by

$$\delta_{\omega,\beta}(v) = -\frac{s_\beta}{r} + 1 + \frac{1}{2}\frac{\mu_{\omega,\beta}(v)^2}{\omega^2} \tag{15}$$

Observe that rescaling ω will rescale $\mu_{\omega,\beta}$ by the same factor, while leaving $\delta_{\omega,\beta}$ invariant. A torsion-free sheaf $F \in \operatorname{Coh} X$ is β-twisted Gieseker stable if for every subsheaf $G \subset F$ we have

$$\begin{aligned} &\mu_{\omega,\beta}(G) \leq \mu_{\omega,\beta}(F), \quad \text{and} \\ &\mu_{\omega,\beta}(G) = \mu_{\omega,\beta}(F) \Rightarrow \delta_{\omega,\beta}(G) > \delta_{\omega,\beta}(F). \end{aligned}$$

Combining the Hodge Index theorem with the assumption $v^2 \geq -2$ shows

$$\delta_{\omega,\beta}(v) \geq -\frac{s_\beta}{r} + 1 + \frac{c_\beta^2}{2r^2} = \frac{v^2+2}{2r^2} + \left(1 - \frac{1}{r^2}\right) \geq 0. \tag{16}$$

Given a class v with $r > 0$, we can write the central charge of equation (9) as

$$\frac{1}{r}Z_{\omega,\beta}(v) = i\mu_{\omega,\beta}(v) + \frac{\omega^2}{2} - \frac{s_\beta}{r} = i\mu_{\omega,\beta}(v) + \frac{\omega^2}{2} - 1 - \frac{\mu_{\omega,\beta}(v)^2}{2\omega^2} + \delta_{\omega,\beta}(v) \tag{17}$$

We now fix a class $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$ with $r(v) > 0$ and $\mu_{\omega,\beta}(v) > 0$.

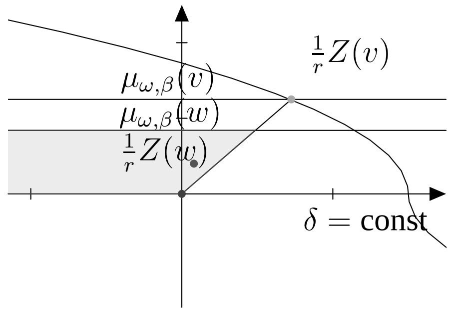

FIGURE 1. Destabilizing subobjects must have smaller δ

Lemma 8.10. *Assume $\omega^2 > 2$. Any class $w \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ with $r(w) > 0$, $0 < \mu_{\omega,\beta}(w) < \mu_{\omega,\beta}(v)$ such that the phase of $Z_{\omega,\beta}(w)$ is bigger or equal to the phase of $Z_{\omega,\beta}(v)$ satisfies $\delta_{\omega,\beta}(w) < \delta_{\omega,\beta}(v)$.*

Proof. By equation (17), it is evident that decreasing $\delta_{\omega,\beta}(v)$ while keeping $\mu_{\omega,\beta}(v)$ fixed will increase the phase of the complex number $Z_{\omega,\beta}(v)$. The same equation also shows that objects with fixed $\delta_{\omega,\beta}$ lie on a parabola, symmetric to the real axis, which intersects the positive real axis; in particular, increasing $\mu_{\omega,\beta}(v)$ while keeping $\delta_{\omega,\beta}(v)$ fixed will also increase the phase of $Z_{\omega,\beta}(v)$; see also fig. 1. □

Definition 8.11. Define $D_v \subset H^*_{\mathrm{alg}}(X, \mathbb{Z})$ as the subset

$$\left\{w : \ 0 < r(w) \leq r(v), w^2 \geq -2, 0 < \mu_{\omega,\beta}(w) < \mu_{\omega,\beta}(v), \delta_{\omega,\beta}(w) < \delta_{\omega,\beta}(v)\right\}.$$

The set D_v is finite: the Hodge Index theorem and $r(w)^2\delta_{\omega,\beta}(w) < r(v)^2\delta_{\omega,\beta}(v)$ bound the norm of the orthogonal projection of $c_\beta(w)$ to $\omega^\perp \subset H^{1,1}_{\mathrm{alg}}(X)_{\mathbb{R}}$; the inequality $0 < c_\beta(w) < r(v)c_\beta(v)$ bounds the projection of $c_\beta(w)$ to $\mathbb{R} \cdot v$; and, finally, $w^2 \geq -2$ and $\delta_{\omega,\beta}(w) < \delta_{\omega,\beta}(w)$ give bounds for $s_\beta(w)$. We also observe that D_v does not change when we rescale ω in the ray $\mathbb{R}_{>0} \cdot \omega$.

Definition 8.12. We define $\mu^{\max}(v)$ by

$$\mu^{\max}(v) := \max\left\{\mu_{\omega,\beta}(w) : \ w \in D_v\right\} \cup \left\{\frac{r(v)}{r(v)+1} \cdot \mu_{\omega,\beta}(v)\right\}.$$

Lemma 8.13. *Let $\mathcal{E}$ be a β-twisted Gieseker stable sheaf $\mathcal{E}$ with $v(\mathcal{E}) = v$. If $\omega^2 > 2 + \frac{2\mu^{\max}(v)}{\mu_{\omega,\beta}(v) - \mu^{\max}(v)}\delta_{\omega,\beta}(v)$, then $\mathcal{E}$ is $Z_{\omega,\beta}$-stable.*

Proof. Consider a destabilizing short exact sequence $A \hookrightarrow \mathcal{E} \twoheadrightarrow B$ in $\mathcal{A}(\omega, \beta)$ with $\phi_{\omega,\beta}(A) \geq \phi_{\omega,\beta}(\mathcal{E})$. By the long exact cohomology sequence, A is a sheaf. Consider the HN-filtration of A with respect to $\mu_{\omega,\beta}$-slope stability in $\mathrm{Coh}\, X$, and let $A_1, \dots, A_n$ be its HN-filtration factors. Since $A \in \mathcal{A}(\omega, \beta)$ we have $\mu_{\omega,\beta}(A_i) > 0$ for all i. Since the kernel of $A \to E$ lies in $\mathcal{F}(\omega, \beta)$, we also have $\mu_{\omega,\beta}(A_i) \leq \mu_{\omega,\beta}(A_1) \leq \mu_{\omega,\beta}(v)$.

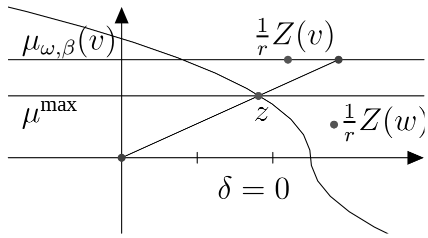

FIGURE 2. Phases of $Z(w)$, z and $Z(v)$

By the see-saw property, we can choose an i such that $\phi_{\omega,\beta}(A_i) \geq \phi_{\omega,\beta}(v)$.

First assume $\mu_{\omega,\beta}(A_i) = \mu_{\omega,\beta}(v)$, in which case $i = 1$. Consider the composition $g\colon A_1 \hookrightarrow A \to \mathcal{E}$ in $\operatorname{Coh} X$. If g is not injective, then $\ker g$ has the same slope $\mu_{\omega,\beta}(\ker g) = \mu_{\omega,\beta}(v)$. Since $\ker g \hookrightarrow A$ factors via $\mathcal{H}^{-1}(B) \hookrightarrow A$, this is a contradiction to $\mathcal{H}^{-1}(B) \in \mathcal{F}(\omega, \beta)$. However, if g is injective, $A_1 \subset \mathcal{E}$ is a subsheaf with $\mu_{\omega,\beta}(A_1) = \mu_{\omega,\beta}(\mathcal{E})$ and, by assumption and equation (17), $\delta_{\omega,\beta}(A_1) \leq \delta_{\omega,\beta}(\mathcal{E})$. This contradicts the assumption that $\mathcal{E}$ is β-twisted Gieseker stable.

We have thus proved $\mu_{\omega,\beta}(A_i) < \mu_{\omega,\beta}(v)$. Let $w \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be the primitive class such that $v(A_i)$ is a positive integer multiple of w. We claim that in fact $\mu_{\omega,\beta}(w) = \mu_{\omega,\beta}(A_i) \leq \mu^{\max}(v)$. In case $r(w) \leq r(v)$, this follows from Lemma 8.10 and the definition of the set D_v. In case $r(w) \geq r(v) + 1$, we observe

$$\omega.c_\beta(w) \leq \omega.c_\beta(A_i) = \Im Z(A_i) \leq \Im Z(A) \leq \Im Z(\mathcal{E}) = \omega.c_\beta(v)$$

to conclude $\mu_{\omega,\beta}(w) \leq \frac{r(v)}{r(v)+1} \cdot \mu_{\omega,\beta}(v)$.

We conclude the proof with a simple geometric argument, see also fig. 2: By equation (17), the phase of $Z(w)$ is less than or equal to the phase $\phi(z)$ of

$$z := i\mu^{\max}(v) + \frac{\omega^2}{2} - 1 - \frac{\mu^{\max}(v)^2}{2\omega^2}.$$

We have $\Im \frac{\mu_{\omega,\beta}(v)}{\mu^{\max}(v)} z = \Im \frac{1}{r(v)} Z(v)$ and

$$\begin{aligned}\Re \frac{\mu_{\omega,\beta}(v)}{\mu^{\max}(v)} z &= \frac{\omega^2}{2} - 1 + \frac{\mu_{\omega,\beta}(v) - \mu^{\max}(v)}{\mu^{\max}(v)} \left(\frac{\omega^2}{2} - 1\right) - \frac{\mu_{\omega,\beta}(v)\mu^{\max}(v)}{2\omega^2} \\ &> \frac{\omega^2}{2} - 1 - \frac{\mu_{\omega,\beta}(v)^2}{2\omega^2} + \delta_{\omega,\beta}(v) = \Re \frac{1}{r(v)} Z(v).\end{aligned}$$

and thus $\phi(z) < \phi_{\omega,\beta}(v)$. This leads to the contradiction

$$\phi_{\omega,\beta}(\mathcal{E}) \leq \phi_{\omega,\beta}(A_i) \leq \phi(z) < \phi_{\omega,\beta}(\mathcal{E}).$$

$\square$

Corollary 8.14. *Let $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$ be a primitive Mukai vector with $v^2 \geq 2$. Let $\omega, \beta \in \mathrm{NS}(X,\mathbb{R})$ be generic with respect to v, and such that $r(v) > 0$ and $\omega.c_\beta(v) > 0$. Let $M_\omega(v)$ be the moduli space of β-twisted Gieseker stable sheaves. If $\mu^{\max}(v)$ is as given in Definition 8.12 and ω satisfies $\omega^2 > 2 + \frac{2\mu^{\max}(v)}{\mu_{\omega,\beta}(v)-\mu^{\max}(v)}\delta_{\omega,\beta}(v)$, then*

$$\theta_v(w_{\sigma,\beta}) \subset \mathrm{Amp}(M_H(v)).$$

This bound does not depend on the specific K3 surface X, other than on its lattice $H^*_{\mathrm{alg}}(X,\mathbb{Z})$. We could also make it completely independent of X by considering the full lattice $H^*(X,\mathbb{Z})$ instead of $H^*_{\mathrm{alg}}(X,\mathbb{Z})$ in Definition 8.11.

9. Hilbert scheme of points on K3 surfaces

In this section, we study the behavior of our nef divisor at walls for the Hilbert scheme of points on a K3 surface. Its walls in $\mathrm{Stab}(X)$ have been described partly in [ABL07, section 3], and the arguments are identical as in the case of $\mathbb{P}^2$ treated in [ABCH12].

Let X be a K3 surface surface with $\mathrm{Pic}(X) = \mathbb{Z}\cdot H$, where H is an ample line bundle. We set $H^2 = 2d$ for some $d \in \mathbb{Z}, d \geq 1$. As in the previous section, we write $\sigma_{t,b} = \sigma_{\omega,\beta}$ for for $\omega = tH, t > 0$ and $\beta = bH$. Let $v = (1, 0, 1-n)$. Then for $t \gg 0$ and $b < 0$ we have $M_{t,b}(v) := M_{\sigma_{t,b}}(v) = \mathrm{Hilb}^n(X)$. We will denote by $\tilde{H} \subset \mathrm{Hilb}^n(X)$ the divisor of subschemes intersecting a given curve C in the linear system $|H|$, and by $2B \subset \mathrm{Hilb}^n(X)$ be the divisor of non-reduced subschemes; $\tilde{H}$ and B are a basis for $\mathrm{NS}(\mathrm{Hilb}^n(X))$.

Example 9.1. One wall of the Gieseker chamber is always $b = 0$: In this case $Z(I_Y) \in \mathbb{R}_{>0}$ for all $I_Y \in \mathrm{Hilb}^n(X)$, and the following short exact sequence in $\mathcal{P}(0)$ makes I_Y strictly semistable:

$$0 \to \mathcal{O}_Y[-1] \to I_Y \to \mathcal{O}_X \to 0$$

Further, considering the filtrations of $\mathcal{O}_Y, \mathcal{O}_{Y'}$ by skyscraper sheaves of points, we see that I_Y and $I_{Y'}$ are S-equivalent if and only if Y, Y' define the same point in the Chow variety. It follows that the corresponding nef divisor $l_{t,0}$ contracts exactly the curves that are contracted by the Hilbert-Chow morphism, and $l_{t,0} \sim \tilde{H}$.

If $\sigma_{t,+\epsilon}$ is a stability condition across the Hilbert-to-Chow wall, then the moduli space $M_{t,+\epsilon}(v) \cong \mathrm{Hilb}^n(X)$ is unchanged, but the universal family is changed: the object I_Y is replaced its derived dual $\mathbf{R}\mathcal{H}om(O_X, I_Y)[1]$. This change affects the map ℓ in such a way that the image of a path crossing the Hilbert-to-Chow wall in $\mathrm{Stab}(X)$ will bounce back into the ample cone once it reaches the ray $[\tilde{H}] \subset \mathrm{NS}(\mathrm{Hilb}^n(X))$.

Example 9.2. Next, we consider the path $\sigma_{t,b}$ with $b = -1-\epsilon$ as $t \in (0, +\infty)$ varies; here $\epsilon > 0$ is fixed and sufficiently small, and such that there exists no spherical object U with $\Im Z_{t,b}(U) = 0$. Let t_0 be such that $Z_{t_0,b}(\mathcal{O}(-H))$, and $Z_{t_0,b}(v)$ have the same phase. A

direct computation shows $t_0 = \sqrt{\frac{1}{d}} + O(\epsilon)$, and that for $t > t_0$, the phase of $Z_{t,b}(\mathcal{O}(-H))$ is bigger than the phase of $Z_{t,b}(v)$.

We claim that for $t > t_0$, any $I_Y \in \mathrm{Hilb}^n(X)$ is stable. Consider any destabilizing subobject $A \subset I_Y$ in $\mathcal{A}_{t,b} = \mathcal{A}(\omega = H, \beta = -H - \epsilon H)$; by the long exact cohomology sequence, A is a torsion-free sheaf. As in the proof of Lemma 8.13, consider any slope-semistable sheaf A_i appearing in the HN-filtration of A with respect to ordinary slope-stability. If we write $v(A_i) = (r, cH, s)$, we have $r > 0$ and

$$\Im Z_{t,b}(A_i) = t \cdot 2d \cdot \bigl(c + r + r\epsilon\bigr)$$

We must have

$$\Im Z_{t,b}(A_i) < \Im Z_{t,b}(I_Y) = t \cdot 2d \cdot (1 + \epsilon)$$

and hence $c + r < 1$, or $c \leq -r$. This implies

$$\Im Z_{t,b}(A_i) \leq t \cdot 2d \cdot r\epsilon.$$

Unless $A_i \cong O(-H)^{\oplus r}$, the stable factors of A_i have rank at least two, and thus $\delta_{t,b}(A_i) \geq \frac{3}{4}$, where $\delta_{t,b} = \delta_{tH,bH}$ is given in Definition 8.9. The same computation that lead to equation (17) then shows

$$\Re Z_{t,b}(A_i) \geq t^2 d - 1 + \frac{3}{4} + O(\epsilon).$$

This shows that for $t > t_0$, the object A_i has strictly smaller phase than I_Y, and so I_Y is stable for $t > t_0$.

On the other hand, whenever there is a curve $C \in |H|$ containing Y, the short exact sequence

$$0 \to \mathcal{O}(-H) \to I_Y \to \mathcal{O}_C(-Y) \to 0$$

will make I_Y strictly semistable for $t = t_0$.

This wall is totally semistable when such a curve exists for any Y, that is if and only if $n \leq h^0(\mathcal{O}(H)) = d + 1$. To determine whether this is a fake wall or corresponds to a wall in the nef cone, we would have to determine whether there exists a curve of S-equivalent objects. Rather than answering this question in general, we just observe that if there exists a curve $C \in |H|$ for which there exists degree n morphism $g_n^1 \colon C \to \mathbb{P}^1$, then the pullbacks $(g_n^1)^*(-x)$ for $x \in \mathbb{P}^1$ give a family of subschemes $Y \subset C$ for which $\mathcal{O}_C(-Y)$ is constant. Thus the corresponding family of ideal sheaves I_Y have a filtration into semistable factors with constant filtration quotients, and hence they are S-equivalent.

By [Laz86, Corollary 1.4] and general Brill-Noether theory, a g_n^1 exists for any smooth curve $C \in |H|$ if and only if $n \geq \frac{d+3}{2}$.

Proposition 9.3. *Let X be a K3 surface with $\operatorname{Pic} X = \mathbb{Z} \cdot H$, and $n \geq \frac{d+3}{2}$. The nef cone of $\mathrm{Hilb}^n(X)$ is generated by $\tilde{H}$ and $\tilde{H} - \frac{2d}{d+n} B$.*

Proof. The previous discussion shows that under the assumptions, the nef divisor $\ell_{t_0,b}$ contracts curves, so it lies on the boundary of the nef cone. To determine its class, let $R \cong \mathbb{P}^1$ be one of the contracted curves. Since C intersects a general element in $|H|$ in $2d$ points, and since the linear system given by R will vanish at each point exactly once, we have $R.\tilde{H} = 2d$. On the other hand, $R \cap 2B$ is the ramification divisor of the map $g_n^1 \colon C \to \mathbb{P}^1$; the Riemann-Hurwitz formula gives $R.2B = 2d + 2n$. Since $R.\ell_{\sigma_0} = 0$, this implies the claim. $\square$

For n sufficiently big, the same result could possibly be obtained using the technique of k-very ample vector bundles; see [ABCH12, Section 3] and references therein.

The recent preprint [CK12] discusses the existence of Brill-Noether divisors on normalizations of curves with δ nodes in the linear system $|H|$; the authors conjecture that these produce extremal curves of the Mori cone, see [CK12, Conjecture 8.7]. The above Proposition proves their conjecture in the case $\delta = 0$; hence it would be interesting to see whether Brill-Noether divisors on nodal curves could also produce curves of S-equivalent objects for different walls in $\mathrm{Stab}(X)$ (in the case $n < \frac{d+3}{2}$).

Remark 9.4. The Beauville-Bogomolov form, along with the pairing $N_1(\mathrm{Hilb}^n(X)) \times N^1(\mathrm{Hilb}^n(X)) \to \mathbb{R}$, induces an isomorphism $N_1 \cong (N^1)^\vee \cong N^1$. Since $\theta_v(0, -H, 0) = \tilde{H}$ and $\theta_v(1, 0, n-1) = -B$, the Beauville-Bogomolov pairing is determined by

$$\tilde{H}^2 = 2d, \quad B^2 = -2n+2, \quad \text{and} \quad (H, B) = 0.$$

(See also [HT10, Section 1].) Thus, the computation in the proof of Proposition 9.3 shows that the isomorphism identifies R with $\tilde{H} + \frac{d+n}{2n-2}B$. So the self intersection of R (with respect to the Beauville-Bogomolov form) is given by

$$(R, R) = \tilde{H}^2 + \left(\frac{d+n}{2n-2}\right)^2 B^2 = 2d - \frac{(d+n)^2}{2n-2} = -\frac{n+3}{2} + \frac{(d+1)(2n-d-3)}{2n-2}.$$

As pointed out to us by Eyal Markman, this does not seem fully consistent with a conjectural description of the Mori cone by Hassett and Tschinkel, see [HT10, Conjecture 1.2]. While $n \geq \frac{d+3}{2}$ implies $(R, R) \geq -\frac{n+3}{2}$ in accordance with their conjecture, in general the Mori cone is smaller than predicted. Let h, b be the primitive curve classes on the rays dual to $\tilde{H}, B$; they are characterized by $h.\tilde{H} = 2d, b.B = 1$ and $h.B = b.\tilde{H} = 0$. Our extremal curve is give by $R = h + (d+n)b$. If we let $\overline{R} = h + (d+n+1)b$, then

$$\begin{gathered}(\overline{R}, \overline{R}) = 2d - \frac{(d+n+1)^2}{2n-2} > -\frac{n+3}{2} \quad \text{for } n \gg 0 \\ \overline{R}.(\tilde{H} - \epsilon B) > 0.\end{gathered}$$

However, since $\overline{R}.\left(\tilde{H} - \frac{2d}{d+n}B\right) < 0$, the class $\overline{R}$ cannot be contained in the Mori cone, in contradiction to [HT10, Conjecture 1.2]. The smallest example is the case $d = 2$ and $n = 5$ and $\overline{R} = h + 8b$, which had been obtained earlier by Markman, [Mar12].

Example 9.5. The previous example considered the case where n is large compared to the genus. Let us now consider a case where the number of points is small compared to the genus, where $d = k^2(n-1)$ for some integers $k \geq 2$.

With the notation as in the previous examples, we now consider the path of of stability conditions $\sigma_{t,-\frac{1}{k}}$ for $t > 0$. Then we are in the situation of Example 8.7 with $A = k$ and $B = -1$; more specifically, since $\frac{d+1}{k}$ is not an integer, we are in Case 1, and the moduli space $M_{\sigma_{t,-\frac{1}{k}}}(v)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^n(X)$ for all $t > 0$. Markushevich and Sawon proved the following result:

Theorem 9.6 ([Mar06, Saw07])**.** *Let X be a K3 surface with $\operatorname{Pic} X = \mathbb{Z} \cdot H$, $H^2 = 2d$, and $d = k^2(n-1)$ for integers $k \geq 2, n$.*

(a) $\mathrm{Nef}(\mathrm{Hilb}^n(X))$ *is generated by*

$$\theta(w_{\infty \cdot H,-1/k}) = \theta(0,-H,0) = \tilde{H} \quad \text{and} \quad \theta(w_{0\cdot H,-1/k}) = \theta(k,-H,(n-1)k) = \tilde{H} - kB.$$

(b) All nef divisors are semi-ample. The morphism induced by $w_{\infty\cdot H,-1/k}$ is the Hilbert-to-Chow morphism, while the one induced by $w_{0\cdot H,-1/k}$ is a Lagrangian fibration.

The fact that $w_{\infty\cdot H,-1/k}$ induces the Hilbert-to-Chow morphism follows in our setup simply from the equality $w_{\infty\cdot H,-1/k} = w_{\sigma_{t\cdot H,0}}$ and Example 9.1.

To reprove the existence of the Lagrangian fibration, we can proceed exactly as in [Saw07], except that Bridgeland stability guides and simplifies the arguments: indeed, since $w_{0\cdot H,-\frac{1}{k}}^2 = 0$, Lemma 6.2 shows that the moduli space $Y := M_{\sigma_{t,-\frac{1}{k}}}(-w_{0\cdot H,-1/k})$ is a smooth K3 surface. Let Φ denote the induced Fourier-Mukai transform $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y,\alpha)$; then $\Phi(v)$ has rank 0. Since $Z_{t,-\frac{1}{k}}(-w_{0\cdot H,-1/k}) \in \mathbb{R}_{<0}$, of points on Y are $\Phi(\sigma_{t,-\frac{1}{k}})$-stable of phase 1; hence the stability condition $\Phi(\sigma_{t,-\frac{1}{k}})$ is again of the form $\sigma_{\widehat{H}_t,\widehat{\beta}_t}$ constructed at the beginning of Section 5, up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$. Since $\Im Z_{tH,-\frac{1}{k}H}(v)$ is minimal, this has to be true also for $\Im Z_{\widehat{H}_t,\widehat{\beta}_t}(\Phi(v))$. But then, since the rank of $\Phi(v)$ is zero, this implies that $\widehat{\beta}_t = \widehat{\beta}$ is constant in t and $\widehat{H}_t = u(t)\widehat{H}$, for some function $u(t)$. We claim that for $t \mapsto 0$, we have $u(t) \mapsto \infty$. Indeed, for $t \mapsto 0$, we must have either $u(t) \mapsto 0$, or $u(t) \mapsto \infty$, and if $u(t) \mapsto 0$, then, by Lemma 8.2, $w_{0\cdot\widehat{H},\widehat{\beta}} \neq (0,0,1)$, which is a contradiction.

Hence, for $t \mapsto 0$, we are via Φ in the Gieseker chamber of Y. It follows that the moduli space $M_{\sigma_{t,b}}(v) \cong M_{\Phi(\sigma_{t,b})}(\Phi(v))$ is isomorphic to a moduli space of twisted Gieseker

stable sheaves of rank 0; as is well-known and discussed in Example 8.3, the latter admits a Lagrangian fibration.

Example 9.7. The Hilbert scheme of n points admits a divisor D with $q(D) = 0$ if and only if $h^2 d = k^2(n-1)$ for integers h, k. The "Tyurin-Bogomolov-Hassett-Tschinkel-Huybrechts-Sawon conjecture" would imply that in this case, the Hilbert scheme admits a birational model with a Lagrangian fibration; we refer to [Ver10] for some discussion of the history of the conjecture, and [Bea10] for some context.

We now consider the first unknown case:

Theorem 9.8. *Let X be a K3 surface with $\operatorname{Pic} X = \mathbb{Z} \cdot H$ and $H^2 = 2d$. Assume that there is an odd integer k with $d = \frac{k^2}{4}(n-1)$ for some integer n. Then:*

(a) The movable cone $\operatorname{Mov}(\operatorname{Hilb}^n(X))$ is generated by

$$\theta(w_{\infty \cdot H, -\frac{2}{k}\cdot H}) = \theta(0, -H, 0) = \tilde{H}, \textit{ and}$$
$$\theta(w_{0 \cdot H, -\frac{2}{k}\cdot H}) = \theta(k, -2\cdot H, (n-1)k) = 2\tilde{H} - kB.$$

(b) The morphism induced by $w_{\infty\cdot H, -2/k}$ is the Hilbert-to-Chow morphism, while the one induced by $w_{0\cdot H,-2/k}$ is a Lagrangian fibration.

(c) All minimal models for $\operatorname{Hilb}^n(X)$ arise as moduli spaces of stable objects in $\mathrm{D}^{\mathrm{b}}(X)$ and their birational transformations are induced by crossing a wall in $\operatorname{Stab}^\dagger(X)$.

Proof. We consider the family of stability conditions $\sigma_{t,-\frac{2}{k}}$ for $t > 0$. As in the previous case, the stability condition $\sigma_{t,-\frac{2}{k}}$ exists for all $t > 0$, since $\frac{4d+1}{k} \notin \mathbb{Z}$. As before, we will study the wall-crossing for the moduli spaces $M_{\sigma_{t,-\frac{2}{d}}}(v)$ for $v = (1, 0, 1-n)$.

Proceeding as in Example 9.5, we consider the smooth projective K3 surface $Y := M_{\sigma_{t,-\frac{2}{k}}}(-w_{0\cdot H,-2/k\cdot H})$ and the induced Fourier-Mukai transform $\Phi\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y, \alpha)$. The assumption $d = \frac{k^2}{4}(n-1)$ implies that $\Phi(v)$ has rank 0, and the same computation as above shows that $t \mapsto 0$ on X corresponds to $t \to +\infty$ via Φ; in particular, for $t \ll 1$ the stability condition $\sigma_{t,-\frac{2}{k}}$ is in the Gieseker chamber of Y for $\Phi(v)$; thus $M_0 := M_{\sigma_{\epsilon,-\frac{2}{k}}}(v)$ admits a Lagrangian fibration, as discussed in Example 8.3.

This also shows that this path meets walls only at finitely many points $t_1, \dots, t_m \in \mathbb{R}$. Denote the moduli stacks of stable objects by $\mathfrak{M}_i := \mathfrak{M}_{\sigma t_i + \epsilon, -\frac{2}{k}}(v)$, and their coarse moduli spaces by M_i. By the results in Section 6, each M_i is a smooth irreducible projective variety for dimension $2n$. We will first prove by descending induction on i that M_i is birational to the Hilbert scheme $\operatorname{Hilb}^n(X) = M_{m+1}$; this will prove claims (a) and (b).

Let $(Z_i, \mathcal{A}) = \sigma_i = \sigma_{t_i,-\frac{2}{k}}$ be a stability condition on one of the walls. We want to show that the set of σ_i-stable objects of class v is non-empty.

Note that $\Im(Z_t) \in \frac{1}{k}2dt \cdot \mathbb{Z}$ and $\Im(Z_t(v)) = \frac{1}{k}4dt$ for all $t \in \mathbb{R}$. Thus, if an object $E \in \mathfrak{M}_{i+1}$ is strictly σ_i-semistable, then it fits into a short exact sequence $A \hookrightarrow E \twoheadrightarrow B$ in $\mathcal{A}$ where A, B are stable of the same phase, and satisfy

$$\Im(Z_i(A)) = \Im(Z_i(B)) = \frac{1}{k}2dt_i. \tag{18}$$

(In fact we have $Z_i(A) = Z_i(B) = \frac{1}{2}Z_i(v)$, and there are only finitely many classes $v(A), v(B)$ that are possible, see e.g. [Tod08, Lemma 3.15].) Mukai's Lemma, see [Muk87, Corallary 2.8] and [Bri08, Lemma 5.2], shows that

$$\mathrm{ext}^1(E,E) \geq \mathrm{ext}^1(A,A) + \mathrm{ext}^1(B,B). \tag{19}$$

Since $A \neq B$ are stable and of the same phase, we have $\mathrm{Hom}(A,B) = 0 = \mathrm{Hom}(B,A) = \mathrm{Ext}^2(A,B)$ and thus $-(v(A),v(B)) = \chi(A,B) = -\mathrm{ext}^1(A,B)$; in particular, $\mathrm{ext}^1(A,B)$ is constant as A, B vary in their moduli space of stable objects. Since A, B, E are simple, the bilinearity of χ then gives

$$2\mathrm{ext}^1(A,B) = 2 + \mathrm{ext}^1(E,E) - \mathrm{ext}^1(A,A) - \mathrm{ext}^1(B,B). \tag{20}$$

We can distinguish two cases:

(a) $(v(A), v(B)) = \mathrm{ext}^1(A,B) = 1$.
(b) $(v(A), v(B)) = \mathrm{ext}^1(A,B) \geq 2$.

In case (b), the point on M_{i+1} corresponding to E lies on a rational curve: by Lemma 8.6, each of the extensions parametrized by $\mathbb{P}\,\mathrm{Ext}^1(A,B)$ is $\sigma_{t_i+\epsilon,-\frac{2}{k}}$-stable. As M_{i+1} is K-trivial, it is not covered by rational curves; this means that a generic object $E \in \mathfrak{M}_{i+1}$ cannot be destabilized by objects of classes $v(A), v(B)$.

We will now proceed to show that case (a) cannot appear. This will imply that there is an open subset $U \subset \mathfrak{M}_{i+1}$ of objects that are still stable on the wall at σ_i, and thus U gives a common open subset of M_i, M_{i+1}, inducing a birational map.

So assume we are in case (a). Then $\mathrm{ext}^1(A', B') = \mathbb{C}$ will hold for any pair $A' \in \mathfrak{M}_{\sigma_i}(v(A))$, $B' \in \mathfrak{M}_{\sigma_i}(v(B))$. By Lemma 8.6, the objects E' corresponding to the unique extension $A' \hookrightarrow E' \twoheadrightarrow B'$ will be $\sigma_{t_i+\epsilon,-\frac{2}{k}}$-stable; thus the universal family of extensions over $\mathfrak{M}_{\sigma_i}(v(A)) \times \mathfrak{M}_{\sigma_i}(v(B))$ induces an injective morphism

$$M_{\sigma_i}(v(A)) \times M_{\sigma_i}(v(B)) \hookrightarrow M_{i+1}$$

However, equations (19) and (20) also show that in case (a) we have $\mathrm{ext}^1(E,E) = \mathrm{ext}^1(A,A) + \mathrm{ext}^1(B,B)$, in other words the above morphism is an injective morphism between projective varieties of the same dimension. As each M_{i+1} is an irreducible holomorphic symplectic variety, this is a contraction unless one of the moduli spaces on the left is a point, i.e., unless A or B is a spherical object.

Let ξ be the Mukai vector of this spherical object. Then $\xi^2 = -2$ and the assumption of case (a) implies $(v - \xi, \xi) = 1$, and so $(v, \xi) = -1$. If $u = (0, \frac{H}{k}, 1-n)$, then

$\Im Z_t(\_) = t(u, \_)$, and so equation (18) implies $(u, \xi) = \frac{1}{2}(u, v) = \frac{1}{2}(n-1)$. By Lemma 9.9, such a class $\xi \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ does not exist. This finishes the proof of claims (a) and (b).

It remains to prove claim (c). By the previous part, wall-crossing induces a chain of birational maps

$$M_0 \dashrightarrow M_1 \dashrightarrow \cdots \dashrightarrow M_n = \mathrm{Hilb}^n(X)$$

As each M_i is a smooth K-trivial variety, the moduli spaces are isomorphic outside of codimension two, and thus we can canonically identify their Néron-Severi groups:

$$\mathrm{NS}(M_0) = \mathrm{NS}(M_1) = \cdots = \mathrm{NS}(\mathrm{Hilb}^n(X)).$$

Theorem 4.1 produces maps $l_i \colon I_i \to N^1(\mathrm{Hilb}^n(X))$, where $I_0 = (0, t_0]$, $I_1 = [t_0, t_1]$, etc. By Lemma 9.10 below, the maps l_i, l_{i+1} agree on the overlap t_i; hence they produce a continuous path $l \colon (0, +\infty) \to \mathrm{NS}(\mathrm{Hilb}^n(X))$ with starting point at a Lagrangian fibration, and endpoint at a divisorial contraction. Since $\mathrm{Hilb}^n(X)$ has Picard rank two, this path must hit the nef cone of any minimal model of $\mathrm{Hilb}^n(X)$. □

Lemma 9.9. *Let X be K3 surface with assumptions as in Theorem 9.8. Let $u, v \in H^*_{\mathrm{alg}}(X, Z)$ be given by $v = (1, 0, 1-n)$ and $u = (0, \frac{H}{k}, 1-n)$. Then there exists no class $\xi \in H^*_{\mathrm{alg}}(X, Z)$ satisfying the following three equations:*

$$(u, \xi) = \frac{n-1}{2} \quad \textit{and} \quad (v, \xi) = -1 \quad \textit{and} \quad \xi^2 = -2.$$

Proof. Observe that $v^2 = 2n-2$, that $(u, v) = n-1$ and $u^2 = \frac{2d}{k^2} = \frac{1}{2}(n-1)$. Hence the orthogonal complement of u and v is given by $\left(\mathbb{R}\cdot u + \mathbb{R}\cdot v\right)^\perp = \mathbb{R}\cdot(2u - v) \subset H^*_{\mathrm{alg}}(X)_{\mathbb{R}}$. As $\xi_0 = (\frac{1}{2}, 0, \frac{1}{2}(n+1))$ satisfies the linear equations of the Lemma, any solution to the linear equations above is given by $\xi = \xi_0 + \alpha(2u - v)$. Since the rank of ξ must be integral, we have $\alpha \in \frac{1}{2} + \mathbb{Z}$.

Using $\xi_0^2 = -\frac{1}{2}(n+1)$ and $(\xi_0, 2u - v) = n$ as well as $(2u - v)^2 = 0$, we obtain

$$\xi^2 = -2 \Leftrightarrow (4\alpha - 1)\cdot n = -3.$$

As $4\alpha \in \mathbb{Z}$, this is a contradiction to $n \geq 5$. □

Lemma 9.10. *Let X be a projective K3 surface, and let $v \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ be a primitive class with $v^2 \geq -2$. With notation as in Theorem 1.4, assume that there there exists a σ_0-stable objects of class v; identify the Néron-Severi groups of $M_\pm$ by extending the birational morphism $M_+ \dashrightarrow M_-$ induced by the common open subset $M^s_{\sigma_0}(v)$ to an isomorphism outside of codimension two.*

Under this identification, we have an equality $l_{0,+} = l_{0,-}$ in $N^1(M_+) = N^1(M_-)$.

The proof is based on the notion of *elementary modifications* in derived categories, introduced by Abramovich and Polishchuk.

Proof. First note that since $M_\pm$ are K-trivial, there exists a common open subset $V \subset M_\pm$ that contains $M^s_{\sigma_0}$ and has complement of codimension 2; this gives the identification of the Néron-Severi groups.

We can normalize the stability condition $\sigma_0 = (Z_0, \mathcal{P}_0)$ such that $\mathfrak{M}_\pm$ parametrizes families of σ_0-semistable objects of phase 1; we can also assume that σ_0 is algebraic.

Since $M_\pm$ are projective, $l_{0,\pm}$ are determined by their degrees on smooth curves that are contained in V and intersect $M^s_{\sigma_0}(v)$; let $C \subset V$ be such a curve, and let $U \subset C$ be the open subset $C \cap M^s_{\sigma_0}(v)$. Since the Brauer group of a curve is trivial, there exists a universal family $\mathcal{E}_+$ on C, corresponding to a lift $C \to \mathfrak{M}_{\sigma_+}(v)$ of the map $C \hookrightarrow V \hookrightarrow M_+$. Similarly, we obtain a family $\mathcal{E}_-$ via the embedding $C \hookrightarrow V \hookrightarrow M_-$. We can choose $\mathcal{E}_\pm$ such that $\mathcal{E}_+|_U \cong \mathcal{E}_-|_U$. Using further twists by a line bundle, we can assume that this isomorphism extends to a morphism $\phi\colon \mathcal{E}_+ \to \mathcal{E}_-$ (this follows, e.g., from [Lie06, Proposition 2.2.3]).

We can think of $\mathcal{E}_\pm$ as flat families of objects in $\mathcal{P}_0(1)$. It is proved in [AP06, Lemma 4.2.3] that the morphism ϕ can be given by a sequence

$$\mathcal{E}_+ = \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_k = \mathcal{E}_-$$

of "elementary modifications" $\mathcal{E}_{i-1} \to \mathcal{E}_i$: this means that there is a point $c_i \in C - U$, an object $Q_i \in \mathcal{P}_0(1)$ such that $\mathcal{E}_{i-1}$ fits into the short exact sequence

$$\mathcal{E}_{i-1} \hookrightarrow \mathcal{E}_i \twoheadrightarrow (i_{c_i})_* Q_i$$

(This is proved in [AP06, Section 4.2] in the case of a smooth affine curve, and U the complement of a point, but the proof extends directly to our case.)

Since $\Im Z_0(Q_i) = 0$, we obtain

$$l_{0,-}.C = \Im Z_0((p_X)_*\mathcal{E}_-) = \Im Z_0((p_X)_*\mathcal{E}_+) + \sum_i \Im Z_0(Q_i) = l_{0,+}.C$$

as claimed. □

Example 9.11. Finally, let us point out that in the case of the Hilbert scheme of n points on an abelian surface T of Picard rank one, Yanagida-Yoshioka and Meachan (in the case of a principally polarized abelian surface) have obtained examples with an infinite series of walls inside the subset $U(T)$ of "geometric" stability conditions; see [Mea12, Section 4] and [YY12, Section 5 and 6]. Their examples have in common that each series of walls contains an infinite sequence of bouncing walls, corresponding to divisorial contractions. Between two bouncing walls, there may be finitely many flopping walls; the corresponding line bundle l_σ will traverse finitely many birational models of Hilb^n.

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